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Spurious correlation of *I*(1) regressors in models with an *I*(0) dependent variable: asymptotic results

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Spurious correlation of $I(1)$ regressors in models with an $I(0)$ dependent variable: asymptotic results

Chris Stewart[1]

Abstract

Using asymptotic theory we have demonstrated that the t-ratios (F-statistic) on $I(1)$ regressors in a model with an $I(0)$ dependent variable (with or without additional $I(0)$ regressors) converge to a random variable and not zero. Simulation results are cited which suggests that these t- and F-statistics will exhibit spurious correlation thereby indicating that spurious regression is more widespread than previously thought. Given that the specification considered is of the error-correction form (under the null of no cointegration) this spurious correlation is argued to potentially give rise to spurious cointegration. One area for further work is to provide more simulation evidence on the models examined here. This work is underway.

Key words: spurious correlation, asymptotic results, unbalanced regression, spurious cointegration.

JEL classification: C01

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1. Introduction

Spurious regression refers to exaggerated correlation indicated by various statistics in a (typically) linear regression model (LRM). This could be manifested in exaggerated magnitudes of coefficients, overstated absolute values of t-ratios, an inflated R^2, a too low regression standard error and/or excessively large F-statistics. The focus has typically been on the appearance of correlation between independently generated variables as indicated by t- and F-tests for zero restrictions on coefficients.[2] In particular, spurious regression occurs when two or more independently generated variables are regressed upon each other and the coefficients of the regressors are found to be statistically significant more frequently than the specified nominal level of significance. This will be the case if t- or F-test statistics for zero restrictions diverge to infinity asymptotically, because they will generally exceed (in magnitude) their corresponding critical values.

Numerous cases where spurious regression is evident have been demonstrated in the literature and are discussed in section 2. One case that has not been considered is for the regression of an $I(0)$ process on a set of independently generated $I(1)$ regressors (with or without an additional set of independent $I(0)$ explanatory variables). We investigate whether there is spurious correlation in such models for two reasons. Firstly, this will extend the cases where the existence of spurious regression has been examined thereby filling this gap in the literature. Secondly, this specification may be viewed as an error-correction model under the null hypothesis of no cointegration. Anticipating the results of this paper, a finding of spurious significance on the $I(1)$ regressors in such a model would be equivalent to establishing the existence of spurious cointegration. That is, the only reason why the $I(1)$ regressors should be significant is if they form a stationary linear combination and, therefore, represent an error-correction mechanism. Through the Granger Reresentation Theorem this would imply cointegration among the $I(1)$ regressors. However, if these regressors are independently generated they should not be correlated, let alone cointegrated. Hence, establishing spurious significance among these $I(1)$ terms would imply spurious cointegration: finding cointegration with a greater probability than indicated by the nominal size of the (t-) test.[3] Such a finding of spurious cointegration is important because of the widespread use of error-correction models (ECM) in the literature.

This paper is organised as follows. Section 2 reviews the literature on spurious correlation and spurious cointegration. Section 3 derives the asymptotic results for various statistics in regressions of an $I(0)$ dependent variable on $I(1)$ regressors (with and without additional $I(0)$ terms). Section 4 offers some concluding remarks.

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[2] From the point of view of the applied econometrician it is t- and F-tests that are used to determine whether variables are significantly correlated with the dependent variable and thus indicate whether they should remain in the model. Hence, the emphasis is on such t- and F-statistics, rather than particular coefficient values or even the R^2, in terms of whether erroneous inference is being drawn.

[3] Spurious cointegration could also be indicated by non-zero coefficients on the $I(1)$ regressors in the models under consideration.

2. Literature review

Granger and Newbold (1974), Phillips (1986) and Entorf (1997)[4] demonstrate that there will be spurious regression on the slope coefficients when an $I(1)$ variable is regressed upon an intercept and a set of $I(1)$ variables (when all the variables are independently generated with or without drifts) using Ordinary Least Squares (OLS).[5]

Nelson and Kang (1984)[6] and Durlauf and Phillips (1988) show that there will be spurious correlation in OLS regressions of a random walk (with or without drift) on an intercept and a linear time trend. Spurious correlation is also found for both slope coefficients in the regression of a driftless random walk on an intercept, another independent driftless random walk and a linear trend.[7]

When an $I(2)$ variable is regressed on deterministic terms, a set of $I(1)$ series and a set of $I(2)$ regressors using OLS Haldrup (1994) finds that when the error term is either $I(1)$ or $I(2)$ the slope coefficients suffer from spurious regression.

Granger *et al* (2001) find spurious regression between two independent stationary series that follow first-order positive autoregressions (and k^{th}–order moving average processes).[8]

Kim *et al* (2004) find spurious correlation in the OLS regression of the two-variable LRM involving variables generated (independently) by linear time trends with stationary first-order autoregressive [AR(1)] errors (or first-order moving average processes or fractionally integrated processes with the difference parameter within the range ±½) and drifts.[9]

Marmol (1995 and 1996) considers the two- and three-variable LRM with the dependent and independent variables being zero mean white noise processes after being differenced d_1, d_2 and d_3 times, respectively, where d_1, d_2 and d_3 are strictly positive integers.[10] The slope coefficients are shown to exhibit spurious correlation.

Marmol (1998) uses asymptotic theory to consider spurious correlation in regressions involving fractionally integrated nonstationary processes (the order of fractional integration exceeds ½) and deterministic terms. The slope coefficients are all found to be subject to spurious regression (even when a set of explanatory variables have a

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[4] Entorf (1997) also shows that spurious regression afflicts the OLS estimator of the two-variable LRM in a panel context (with fixed effects).

[5] Choi (1994), using asymptotic theory, shows that if any of the regressors are conitegrated in such regressions then their coefficients will not suffer from spurious regression. However, the coefficients of the non-cointegrated regressors will.

[6] Use of the Cochrane-Orcutt estimation procedure reduces the degree of spurious regression but does not remove it for all regressions considered by Nelson and Kang (1984).

[7] An interesting result is that the slope coefficient tends to its expected population value of zero asymptotically while its t-ratio simultaneously diverges to infinity in large samples.

[8] Banerjee *et al* (1993) find, using Monte Carlo simulation, no spurious regression when two independently generated and non-autocorrelated $I(0)$ processes are regressed upon each other in the two-variable LRM.

[9] Hassler (2003) finds spurious regression in models involving stationary processes generated to allow for time-varying means, possibly arising from structural breaks or seasonality.

[10] Marmol (1995) analyses the case where all variables are integrated of the same order and Marmol (1996) the situation where they feature different orders of integration.

lower order of integration than the dependent variable and the other independent variables) provided the regression error term is nonstationary (its order of integration exceeds ½).[11]

Tsay and Chung (2000) examine the possibility of spurious regression in the two-variable LRM for combinations of fractionally integrated processes that include nonstationary, stationary (the order of integration is in the range ±½) and deterministic processes. Spurious regression is found for regressions involving two independent nonstationary processes and two independent stationary processes (provided the sum of the two variables' orders of integration exceeds a half). It is also found in the regression of a nonstationary process on an independent stationary process with long memory (the order of integration exceeds zero),[12] and vice-versa,[13] and the regression of either a nonstationary (or stationary) process on a linear time trend.

Sun (2006) demonstrates that spurious correlation can occur in regressions involving two stationary generalised fractionally integrated processes – in particular, Gegenbauer processes are considered.[14] It is concluded that spurious regression arises when the sum of the processes' generalised differencing parameter exceeds 0.5 and their spectral densities have poles at the same location.

Hassler (1996), using asymptotic theory argues that there is no spurious significance for the $I(0)$ regressors in a regression of an $I(1)$ process on a set of $I(0)$ variables and $I(1)$ terms. This is based upon the result that the t-ratios on the $I(0)$ regressors converge to a random variable in the probability limit. However, Stewart (2006) argues that there may be spurious significance if the mean of this t-ratio tends to a high value relative to the critical value. Indeed, Monte Carlo simulations show that, when the $I(0)$ regressor is autocorrelated, there is significant spurious regression (the empirical size of the t-ratio exceeds the nominal size used). Further, the degree of spurious correlation rises as the degree of autocorrelation in the $I(0)$ series increases.

A review of the literature suggests that spurious regression occurs in models involving a wide range of combinations of integrated, fractionally integrated, stationary and deterministic variables. However, there has been no consideration of spurious correlation using asymptotic theory in a model with an $I(0)$ dependent variable and a set of $I(1)$ regressors (with or without a set of $I(0)$ explanatory factors).[15] This paper intends to provide such an analysis.

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[11] This is true for the coefficients of deterministic and non-deterministic components.

[12] Tsay and Chung (2000) note that although the intercept and slope coefficient estimators may converge in probability to zero (when the dependent variable is stationary) their corresponding t-ratios always diverge. Hence, it is the divergence of t-ratios that is the form of spurious regression in this case.

[13] There is no spurious regression when the independent variable is a short memory stationary series (the order of integration is between 0 and –½).

[14] A Gegenbauer process generalises the standard fractionally integrated process in that it not only captures long-range dependence (memory), as do standard fractional process, but it can also depict periodic cyclical patterns.

[15] Banerjee *et al* (1993, p. 79) report a Monte Carlo simulation experiment with 100 observations and 100 replications for the regression of a stationary white noise variable on an independently generated pure $I(1)$ process (that is Gaussian after first differencing) in a two-variable LRM and find the empirical size of the t-ratio on the slope coefficient to be 0.0458 (using a 5% nominal size). Thus, there is no evident spurious correlation. They do not consider a situation where there is autocorrelation in

Further, because an error-correction model under the null of no cointegration is a particular case of the form of model we wish to examine spurious correlation may also give rise to spurious cointegration. For example, Ericsson and MacKinnon (2002) discuss a test for cointegration based upon the t-ratio (of some function) of the lagged level of the dependent variable in such an error-correction model. This specification has not been previously investigated in the literature on spurious cointegration. The limited simulation evidence on such error-correction models within the context of spurious cointegration is briefly discussed prior to consideration of the asymptotic theory in this specification.

Using Monte Carlo simulation Franses (1990) finds evidence of spurious cointegration in a single equation autoregressive distributed lag model between two variables using the method discussed in Boswijk (1989).[16] That is, the null of no cointegration is rejected more frequently than it should be.[17]

Leybourne and Newbold (2003) assess the rejection frequencies for a similar cointegration test, see Banerjee *et al* (1986), when the true DGPs are independent integrated series with a single structural break, being either a shift in the intercept or a shift in the slope. They find evidence of serious to severe spurious rejections of the no cointegration null for a sizeable number of the combinations of break points in the two series.[18]

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either the $I(0)$ or $I(1)$ variable. Granger *et al* (2001, p. 901) provide simulation results (in Table 2) for the regression of an autocorrelated stationary series (with autocorrelation coefficient equal to 0.5) on an intercept and a pure $I(1)$ process. They report empirical sizes of the t-ratio on the $I(1)$ regressor (using a 5% nominal size) as 24.8%, 26.2% and 22.8% for sample sizes of 100, 500 and 2000, respectively. However, they do not comment on these simulation results: they are an externality in their analysis of spurious correlation between stationary series.

[16] Two data generation processes (DGPs) are considered for the difference of the dependent variable, an AR(1) process augmented with the difference of the independent variable and a first-order moving average process [MA(1)]. The independent variable is a pure $I(1)$ process (that is, Gaussian after first differencing).

[17] Gonzalo and Lee (1998) define spurious cointegration, within the context of the Johansen procedure, as the LR statistics diverging to infinity with probability one – “a size pitfall”. This is distinguished from size distortion arising from the use of incorrect critical values (for example, due to the incorrect specification of deterministic terms in the system). They find spurious cointegration from the Maximum Eigenvalue and Trace Likelihood Ratio (LR) tests (associated with the Johansen procedure) for the null hypothesis of no cointegration against the alternative of one cointegrating vector in a two-variable system when at least one series is not a pure $I(1)$ process. In particular, when one series is a first-order autoregression with autocorrelation coefficient above one, when both series are generated according to stochastic unit root processes and if at least one of the variables in the Vector Error Correction Model (VECM) is $I(2)$ and there are insufficient lags. However, if the lag length of the VECM is increased there will only be size distortion due to the use of inappropriate critical values. For series with fractionally integrated DGPs (with the differencing parameter greater than 0.5 and not equal to unity) the Johansen LR-statistics exhibit large size distortions, especially when the number of lags employed in the VECM is small – also see Gonzalo and Lee (2000). Spurious cointegration is also demonstrated to arise if deterministic components are not appropriately accounted for in the VECM and if the error covariance matrix of the VECM is (near) singular. The Engle and Granger (EG) method is shown to be more far more robust than the Johansen method (in terms of size distortion) to misspecification of the deterministic components in the model and when the processes’ true DGPs are not $I(1)$. Gonzalo and Lee (1998, p. 146) suggest that if the Johansen and EG methods give different inference in terms of cointegration then this may indicate misspecification of the series’ DGPs as $I(1)$ processes.

[18] Spurious cointegration is also found when the Engle and Granger and Johansen procedures are applied to integrated series with a structural break. The Engle and Granger test is found to be the most

3. Asymptotic results for regressions with an *I*(0) dependent variable and *I*(1) regressors

This section produces the asymptotic results of various statistics in, first, a model with an *I*(0) dependent variable and both *I*(0) and *I*(1) regressors and, second, the regression of an *I*(0) variable on a set of *I*(1) processes.

3.1 Regressions of an I(0) variable on I(0) and I(1) regressors

Consider the regression of a single *I*(0) variable that exhibits first-order autocorrelion [AR(1)], Y_{0t}, on an $m_1 \times 1$ vector of *I*(0) AR(1) processes, $\underline{X}_{1t}$, and an $m_2 \times 1$ vector of *I*(1) variables, $\underline{X}_{2t}$, where a bar below a series denotes a vector (or matrix).[19] That is:

$$Y_{0t} = \hat{a} + \underline{\hat{b}}_1' \underline{X}_{1t} + \underline{\hat{b}}_2' \underline{X}_{2t} + \hat{u}_{1t} \tag{1}$$

with,

$$Y_{0t} = v_{0t} \tag{2}$$

$$\underline{X}_{1t} = \underline{v}_{1t} \tag{3}$$

and,

$$\underline{X}_{2t} = \underline{X}_{2t-1} + \underline{v}_{2t} \tag{4}$$

where v_{0t}, $\underline{v}_{1t}$ and $\underline{v}_{2t}$ are zero mean and *I*(0) such that v_{0t} and the series in the vectors $\underline{v}_{1t}$ and $\underline{v}_{2t}$ may be autocorrelated:[20]

Defining,

$$\underline{w}_t' = \left(v_{0t}, \underline{v}_{1t}', \underline{v}_{2t}' \right) \tag{7}$$

Then the partial sum process can be expressed as:

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robust of the three methods in the sense that it delivers spurious rejections for the fewest number of combinations of break points of the three tests.

[19] The vectors are defined as: $\underline{X}_{1t}' = \left[X_{1t} \quad X_{2t} \quad \ldots \quad X_{m_1 t} \right]$ and $\underline{X}_{2t}' = \left[X_{21t} \quad X_{22t} \quad \ldots \quad X_{2m_2 t} \right]$.

[20] The vectors are defined as: $\underline{v}_{1t}' = \left[v_{1t} \quad v_{2t} \quad \ldots \quad v_{m_1 t} \right]$ and $\underline{v}_{2t}' = \left[v_{21t} \quad v_{22t} \quad \ldots \quad v_{2m_2 t} \right]$.

$$\underline{S}_t = \sum_{j=1}^{t} \underline{w}_j , \qquad t = 1, 2, ..., T \tag{8}$$

Assuming the following multivariate invariance principle after normalisation:

$$T^{-0.5} \underline{S}_{[Tr]} \Rightarrow \underline{B}(r), \qquad r \in [0,1] \tag{9}$$

where $\underline{B}'(r) = \left(B_0(r), \underline{B}_1'(r), \underline{B}_2'(r) \right)$ is a vector of Brownian motion that is partitioned such that $B_0(r), \underline{B}_1'(r)$ and $\underline{B}_2'(r)$ correspond to v_{0t}, $\underline{v}_{1t}$ and $\underline{v}_{2t}$, respectively, and features the covariance matrix:[21]

$$\underline{\Omega} = \lim_{T \to \infty} T^{-1} E\left(\underline{S}_T \underline{S}_T' \right) \tag{10}$$

Following Hassler (1996) it is assumed that v_{0t}, $\underline{v}_{1t}$ and $\underline{v}_{2t}$ are independently generated vectors that are ergodic as well as stationary with the following covariance matrices:[22]

$$p\lim_{T \to \infty} T^{-1} \sum_{t=1}^{T} v_{0t} v_{0t-t} = \Sigma_{0,t} , \qquad t = 0, 1 \tag{11}$$

$$p\lim_{T \to \infty} T^{-1} \sum_{t=1}^{T} \underline{v}_{lt} \underline{v}_{lt-t}' = \underline{\Sigma}_{l,t} , \qquad t = 0, 1 \qquad l = 1, 2 \tag{12}$$

Now denote the variables in equation **(1)** in mean deviation form using lower case letters thus:

$$y_{0t} = Y_{0t} - \overline{Y}_0 , \qquad \overline{Y}_0 = \sum_{t=1}^{T} Y_{0t} \tag{13}$$

$$\underline{x}_{1t} = \underline{X}_{1t} - \overline{\underline{X}}_1 , \qquad \overline{\underline{X}}_1 = \sum_{t=1}^{T} \underline{X}_{1t} \tag{14}$$

$$\underline{x}_{2t} = \underline{X}_{2t} - \overline{\underline{X}}_2 , \qquad \overline{\underline{X}}_2 = \sum_{t=1}^{T} \underline{X}_{2t} \tag{15}$$

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[21] Assumptions on the error terms, v_{0t}, $\underline{v}_{1t}$ and $\underline{v}_{2t}$, to ensure that equation **(9)** holds are given in Phillips and Durlauf (1986, theorem 2.1).

[22] Note that equation **(11)** and equation **(12)** allow for first-order autocorrelation in the error term series. More specific expressions for **(11)** and **(12)** (for v_{0t} and $\underline{v}_{1t}$) are given as equation **(A.1.6)** and **(A.1.15)** in the appendix.

Equation **(1)** can now be re-expressed as:

$$y_{0t} = \underline{\hat{b}}_1' \underline{x}_{1t} + \underline{\hat{b}}_2' \underline{x}_{2t} + \hat{u}_{1t} \tag{16}$$

Similar to Hassler (1996) we collect the following results provided by Phillips (1986, lemma 1) and Park and Phillips (1989, lemma 2.1).[23]

$$T^{-1}\sum_{t=1}^{T} \underline{x}_{1t}\underline{x}_{2t}' \Rightarrow \int_0^1 \underline{B}_2'(r)d\underline{B}_1(r) - \underline{B}_1(r)\int_0^1 \underline{B}_2'(r)dr =: \int \underline{b}_2' d\underline{B}_1 \tag{17}$$

$$T^{-2}\sum_{t=1}^{T} \underline{x}_{2t}\underline{x}_{2t}' \Rightarrow \int_0^1 \underline{B}_2(r)\underline{B}_2' dr - \int_0^1 \underline{B}_2(r)dr\int_0^1 \underline{B}_2'(r)dr =: \int \underline{b}_2\underline{b}_2' \tag{18}$$

$$T^{-1}\sum_{t=1}^{T} \underline{x}_{2t}y_{0t} \Rightarrow \int_0^1 \underline{B}_2(r)dB_0(r) - B_0(r)\int_0^1 \underline{B}_2(r)dr =: \int \underline{b}_2' dB_0 \tag{19}$$

Using **(2)**, **(3)**, **(13)**, **(14)**, **(56)** and **(57)**, gives:[24]

$$\operatorname*{p\lim}_{T\to\infty} T^{-1}\sum_{t=1}^{T} y_{0t}^2 = \operatorname*{p\lim}_{T\to\infty} T^{-1}\sum_{t=1}^{T} v_{0t}^2 = \Sigma_{0,0} \tag{20}$$

$$\operatorname*{p\lim}_{T\to\infty} T^{-1}\sum_{t=1}^{T} \underline{x}_{1t}\underline{x}_{1t-t}' = \operatorname*{p\lim}_{T\to\infty} T^{-1}\sum_{t=1}^{T} \underline{v}_{1t}\underline{v}_{1t-t}' = \underline{\Sigma}_{1,t}, \qquad t = 0,1 \tag{21}$$

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[23] Where the terms converge in distribution as T tends to infinity.

[24] Now $\sum_{t=1}^{T} y_{0t}^2 = \sum_{t=1}^{T}\left(Y_{0t} - \overline{Y}_0\right)^2 = \sum_{t=1}^{T} Y_{0t}^2 - T\left(\overline{Y}_0\right)^2 = T\left(T^{-1}\sum_{t=1}^{T} Y_{0t}^2\right) - \left(T^{0.5}\overline{Y}_0\right)^2$. Using the results from Phillips (1986) and Parks and Phillips (1989) allows the probability limit to be obtained as follows: $p\lim\sum_{t=1}^{T} y_{0t}^2 = T\Sigma_{0,0} - \left[B_0(1)\right]^2 \Rightarrow p\lim T^{-1}\sum_{t=1}^{T} y_{0t}^2 = \Sigma_{0,0}$. Given $\underline{X}_{1t}$ is stationary (constant mean):

$$\sum_{t=1}^{T} \underline{x}_{1t}\underline{x}_{1t-t}' = \sum_{t=1}^{T}\left(\underline{X}_{1t} - \overline{\underline{X}}_1\right)\left(\underline{X}_{1t-t} - \overline{\underline{X}}_{1-t}\right)' = \sum_{t=1}^{T} \underline{X}_{1t}\underline{X}_{1t-t}' - T\overline{\underline{X}}_1\overline{\underline{X}}_{1-t}' - T\overline{\underline{X}}_{1-t}\overline{\underline{X}}_1' + T\overline{\underline{X}}_{1-t}\overline{\underline{X}}_{1-t}'$$

$$\Rightarrow \sum_{t=1}^{T} \underline{x}_{1t}\underline{x}_{1t-t}' = T\left(T^{-1}\sum_{t=1}^{T} \underline{X}_{1t}\underline{X}_{1t-t}'\right) - \left(T^{0.5}\overline{\underline{X}}_1\right)\left(T^{0.5}\overline{\underline{X}}_{1-t}\right)' - \left(T^{0.5}\overline{\underline{X}}_{1-t}\right)\left(T^{0.5}\overline{\underline{X}}_1\right)' + \left(T^{0.5}\overline{\underline{X}}_{1-t}\right)\left(T^{0.5}\overline{\underline{X}}_{1-t}\right)'$$

Using the results from Phillips (1986) and Parks and Phillips (1989) allows the probability limit to be obtained as follows: $p\lim\sum_{t=1}^{T} \underline{x}_{1t}\underline{x}_{1t-t}' = T\underline{\Sigma}_{1,t} - \underline{B}_1(1)\underline{B}_1'(1) - \underline{B}_1(1)\underline{B}_1'(1) + \underline{B}_1(1)\underline{B}_1'(1)$

$$\Rightarrow p\lim T^{-1}\sum_{t=1}^{T} \underline{x}_{1t}\underline{x}_{1t-t}' = \underline{\Sigma}_{1,t}.$$

It can also be shown, assuming that v_{0t} and the series within $\underline{v}_{1t}$ follow stationary first-order autoregressions, that $p\lim\limits_{T\to\infty} T^{-0.5}\sum\limits_{t=1}^{T}\underline{X}_{1t}Y_{0t} = p\lim\limits_{T\to\infty} T^{-0.5}\sum\limits_{t=1}^{T} v_{0t}\underline{v}_{1t} = \underline{\Gamma}$ (a derivation of this result is given in Appendix A1):[25] Combining this with the results provided in Phillips (1986) and Parks and Phillips (1989), gives:[26]

$$p\lim_{T\to\infty} T^{-0.5}\sum_{t=1}^{T}\underline{x}_{1t}y_{0t} = p\lim_{T\to\infty} T^{-0.5}\sum_{t=1}^{T} v_{0t}\underline{v}_{1t} = \underline{\Gamma} \qquad \textbf{(22)}$$

Where, $\underline{\Gamma}$ is a vector of random variables with zero means and non-zero finite variances.

Following Hassler (1996) we apply the formula for inverting partitioned matrices to obtain the following expression for the OLS estimators for equation **(16)**:[27]

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[25] To obtain this probability limit we derive the limit of, respectively, the mean and variance of $T^{d}\sum\limits_{t=1}^{T} v_{0t}\underline{v}_{1t}$. The mean is: $\underset{T\to\infty}{Lim}\ E\left(T^{d}\sum\limits_{t=1}^{T} v_{0t}\underline{v}_{1t}\right) = \underline{0},\ \forall d$. The variance is: $\underset{T\to\infty}{Lim}\ Var\left(T^{d}\sum\limits_{t=1}^{T} v_{0t}\underline{v}_{1t}\right) = \underline{0}$, when $d < -0.5$. For $d = -0.5$, $\underset{T\to\infty}{Lim}\ Var\left(T^{d}\sum\limits_{t=1}^{T} v_{0t}\underline{v}_{1t}\right) = s_{e_0}^{2}\left(\frac{1}{1-r^{2}}\right)\underline{\Omega}$, where, $\underline{\Omega} = \begin{bmatrix} \left[\left(\frac{1}{1-f_1^2}\right)s_{e_1}^{2}\right]\left[\frac{1-r^2f_1^2}{(1-rf_1)^2}\right] & 0 & \dots & 0 \\ 0 & \left[\left(\frac{1}{1-f_2^2}\right)s_{e_2}^{2}\right]\left[\frac{1-r^2f_2^2}{(1-rf_2)^2}\right] & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \left[\left(\frac{1}{1-f_{m_1}^2}\right)s_{e_{m1}}^{2}\right]\left[\frac{1-r^2f_{m_1}^2}{(1-rf_{m_1})^2}\right] \end{bmatrix}$. This is an extension of the results derived in Granger *et al* (2001, p. 901) – our extension is to allow for m_1 $I(0)$ regressors rather than just one. For $d > -\frac{1}{2}$, $\underset{T\to\infty}{Lim}\ Var\left(T^{d}\sum\limits_{t=1}^{T} v_{0t}\underline{v}_{1t}\right) = \begin{bmatrix} \infty & 0 & \dots & 0 \\ 0 & \infty & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \infty \end{bmatrix}$. Thus, $T^{d}\sum\limits_{t=1}^{T} v_{0t}\underline{v}_{1t}$ diverges in the probability limit when $d > -\frac{1}{2}$. The unique value of d that yields a non-zero and finite variance is –0.5, hence giving the exponent of T where the probability limit converges to a random variable (which we denote $\underline{\Gamma}$).

[26] $\sum\limits_{t=1}^{T}\underline{x}_{1t}y_{0t} = \sum\limits_{t=1}^{T}\left(\underline{X}_{1t}-\underline{\overline{X}}_{1}\right)\left(Y_{0t}-\overline{Y}_{0}\right) = \sum\limits_{t=1}^{T}\underline{X}_{1t}Y_{0t} - T\underline{\overline{X}}_{1}\overline{Y}_{0} = T^{0.5}\left(T^{-0.5}\sum\limits_{t=1}^{T}\underline{X}_{1t}Y_{0t}\right) - \left(T^{0.5}\underline{\overline{X}}_{1}\right)\left(T^{0.5}\overline{Y}_{0}\right)$ Using the results from Phillips (1986) and Parks and Phillips (1989) allows the probability limit to be obtained as follows: $p\lim\sum\limits_{t=1}^{T}\underline{x}_{1t}y_{0t} = T^{0.5}\underline{\Gamma} - \underline{B}_{1}(1)B_{0}(1) \Rightarrow p\lim T^{-0.5}\sum\limits_{t=1}^{T}\underline{x}_{1t}y_{0t} = \underline{\Gamma}$.

[27] Defining $\underline{\hat{b}}' = \left[\underline{\hat{b}}_{1} \quad \underline{\hat{b}}_{2}\right]$ and $\underline{x}_{t} = \left[\underline{x}_{1t} \quad \underline{x}_{2t}\right]$ which, when substituted into the OLS estimator, $\underline{\hat{b}} = \left(\underline{x}_{t}'\underline{x}_{t}\right)^{-1}\underline{x}_{t}'y_{0t}$, gives: $\begin{bmatrix}\underline{\hat{b}}_{1} \\ \underline{\hat{b}}_{2}\end{bmatrix} = \left(\begin{bmatrix}\underline{x}_{1t} \\ \underline{x}_{2t}\end{bmatrix}\left[\underline{x}_{1t} \quad \underline{x}_{2t}\right]\right)^{-1}\begin{bmatrix}\underline{x}_{1t} \\ \underline{x}_{2t}\end{bmatrix}y_{0t}$. Expanding this expression yields equation **(23)**.

$$\begin{bmatrix}\hat{\underline{b}}_1\\ \hat{\underline{b}}_2\end{bmatrix}=\begin{bmatrix}A & B\\ C & D\end{bmatrix}\begin{bmatrix}\sum\limits_{t=1}^{T}\underline{x}_{1t}y_{0t}\\ \sum\limits_{t=1}^{T}\underline{x}_{2t}y_{0t}\end{bmatrix} \tag{23}$$

with,

$$A=\left[\sum_{t=1}^{T}\underline{x}_{1t}\underline{x}_{1t}{}'-\sum_{t=1}^{T}\underline{x}_{1t}\underline{x}_{2t}{}'\left(\sum_{t=1}^{T}\underline{x}_{2t}\underline{x}_{2t}{}'\right)^{-1}\sum_{t=1}^{T}\underline{x}_{2t}\underline{x}_{1t}{}'\right]^{-1} \tag{24}$$

$$B=-A\left[\sum_{t=1}^{T}\underline{x}_{1t}\underline{x}_{2t}{}'\left(\sum_{t=1}^{T}\underline{x}_{2t}\underline{x}_{2t}{}'\right)^{-1}\right] \tag{25}$$

$$C=B' \tag{26}$$

$$D=\left(\sum_{t=1}^{T}\underline{x}_{2t}\underline{x}_{2t}{}'\right)^{-1}\left[\underline{I}_{m_2}+\left(\sum_{t=1}^{T}\underline{x}_{2t}\underline{x}_{1t}{}'\right)A\left(\sum_{t=1}^{T}\underline{x}_{1t}\underline{x}_{2t}{}'\right)\left(\sum_{t=1}^{T}\underline{x}_{2t}\underline{x}_{2t}{}'\right)^{-1}\right] \tag{27}$$

where $\underline{I}_{m_2}$ is an $m_2 \times m_2$ identity matrix.

Substitution of **(17)**, **(18)** and **(21)** into **(24)**, after taking the probability limit, gives (proofs of all of the following results are given in Appendix A2):

$$\text{plim}\, TA=\underline{\Sigma}_{1,0}^{-1} \tag{28}$$

Substitution of **(17)**, **(18)** and **(28)** into **(25)**, yields:

$$T^2B\Rightarrow-\underline{\Sigma}_{1,0}^{-1}\int\underline{b}_2{}'\,d\underline{B}_1\left(\int\underline{b}_2\underline{b}_2{}'\right)^{-1} \tag{29}$$

Substitution of **(17)**, **(18)** and **(28)** into **(27)**, gives:

$$T^2D\Rightarrow\left(\int\underline{b}_2\underline{b}_2{}'\right)^{-1} \tag{30}$$

From **(23)** the coefficient vector on the $I(0)$ variables is:[28]

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[28] The results in **(28)**, **(29)** and **(30)** are the same as those reported in Hassler (1996, p. 28, equation 7), because $\underline{x}_{1t}$ and $\underline{x}_{2t}$ are defined as in his paper. However, the limiting distributions for the coefficients

$$\underline{\hat{b}}_1 = A\sum_{t=1}^{T}\underline{x}_{1t}y_{0t} + B\sum_{t=1}^{T}\underline{x}_{2t}y_{0t} \qquad \mathbf{(31)}$$

Substitution of **(19)**, **(22)**, **(28)** and **(29)** in **(31)**, after taking the probability limit, yields the limiting distribution for $\underline{\hat{b}}_1$ thus:

$$p\lim T^{0.5}\underline{\hat{b}}_1 = \underline{\Sigma}_{1,0}^{-1}\underline{\Gamma} =: \underline{\mathbf{b}}_1 \qquad \mathbf{(32)}$$

The coefficient vector $\underline{\hat{b}}_1$ tends to zero in the probability limit. This seems reasonable given that it involves the regression of an *I*(0) variable on a set of independently generated *I*(0) variables.

From **(23)** the coefficient vector on the *I*(1) variables is:

$$\underline{\hat{b}}_2 = C\sum_{t=1}^{T}\underline{x}_{1t}y_{0t} + D\sum_{t=1}^{T}\underline{x}_{2t}y_{0t} \qquad \mathbf{(33)}$$

Substitution of **(19)**, **(22)**, **(29)** and **(30)** into **(33)**, yields the limiting distribution for $\underline{\hat{b}}_2$ thus:

$$T\underline{\hat{b}}_2 \Rightarrow \left(\int \underline{b}_2\underline{b}_2'\right)^{-1}\int \underline{b}_2' dB_0 =: \underline{\mathbf{b}}_2 \qquad \mathbf{(34)}$$

The coefficient vector $\underline{\hat{b}}_2$ tends to zero in the limit. That is, the regression of an *I*(0) variable on a set of independently generated *I*(1) variables yields a zero value for the OLS coefficient estimator in the limit.

The residual variance is given by:[29]

$$s^2 = T^{-1}\sum_{t=1}^{T}\left(y_{0t} - \hat{b}_1'\underline{x}_{1t} - \hat{b}_2'\underline{x}_{2t}\right)^2 \qquad \mathbf{(35)}$$

(and other statistics) are different here from those that he reports because $\underline{y}_{0t}$ is *I*(1) in his article and *I*(0) in this paper.

[29] This derives from the standard formula: $s^2 = T^{-1}\sum_{t=1}^{T}\hat{u}_{1t}^2$. Asymptotically this is the same as the unbiased estimator of the regression standard error.

Substitution of **(17)**, **(18)**, **(19)**, **(20)**, **(21)**, **(22)**, **(32)** and **(34)** into a rearranged form of **(35)**, after taking the probability limit, gives the limiting distribution of s^2, thus:

$$p\lim s^2 = \Sigma_{0,0} \qquad \textbf{(36)}$$

This implies that s^2 converges to a constant. In particular, the variance of the residuals converges to the variance of the demeaned dependent variable. This makes sense given that the coefficients in **(16)** converge to zero in the probability limit implying that $p\lim y_{0t} = \hat{u}_{1t}$.

Following Hassler (1996) we can specify the kth *I*(0) regressor's coefficient standard error as:

$$s_{\left[\underline{\hat{b}}_1\right]_k} = \left(s^2 A_{kk}\right)^{0.5}, \qquad k = 1,2,\dots,m_1 \qquad \textbf{(37)}$$

Substitution of **(28)** and **(36)** into **(37)**, after taking the probability limit, gives:

$$p\lim T^{0.5} s_{\left[\underline{\hat{b}}_1\right]_k} = \left(\Sigma_{0,0}\right)^{0.5}\left(\underline{\Sigma}_{1,0}\right)_{kk}^{-0.5} \qquad \textbf{(38)}$$

This implies that the coefficient standard errors for $\underline{\hat{b}}_1$ converge to zero, in the probability limit, at rate $T^{0.5}$.

The t-ratio for the kth *I*(0) regressor's coefficient is:

$$t_{\left[\underline{\hat{b}}_1\right]_k} = \frac{\left[\underline{\hat{b}}_1\right]_k}{s_{\left[\underline{\hat{b}}_1\right]_k}} \qquad \textbf{(39)}$$

Substitution of **(32)** and **(38)** into **(39)**, after taking the probability limit, gives:

$$p\lim t_{\left[\underline{\hat{b}}_1\right]_k} = \frac{\left[\mathbf{b}_1\right]_k}{\left(\Sigma_{0,0}\right)^{0.5}\left(\underline{\Sigma}_{1,0}\right)_{kk}^{-0.5}} \qquad \textbf{(40)}$$

This implies that the t-ratios for the *I*(0) coefficients converge to random variables, in the probability limit.[30]

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[30] It may seem surprising that the t-ratios of the *I*(0) regressors tend to random variables in the probability limit, given that they were all generated independently of the stationary dependent variable. However, this result is consistent with the findings of Granger *et al* (2001) who find spurious significance (with respect to the t-ratio) in the regression of autocorrelated stationary processes upon each other in the two-variable LRM.

The j$^{\text{th}}$ $I(1)$ variable's coefficient standard error is:

$$s_{[\hat{\underline{b}}_2]_j} = \left(s^2 D_{jj}\right)^{0.5}, \qquad j = 1,2,\ldots,m_2 \tag{41}$$

Substitution of **(30)** and **(36)** into **(41)**, gives:

$$Ts_{[\hat{\underline{b}}_2]_j} \Rightarrow \left(\Sigma_{0,0}\right)^{0.5}\left[\left(\int \underline{b}_2 \underline{b}_2'\right)^{-0.5}\right]_{jj} \tag{42}$$

This implies that the coefficient standard errors for $\hat{\underline{b}}_2$ converge to zero, in the limit, at rate T.

The t-ratio for the j$^{\text{th}}$ $I(1)$ variable's coefficient is:

$$t_{[\hat{\underline{b}}_2]_j} = \frac{[\hat{\underline{b}}_2]_j}{s_{[\hat{\underline{b}}_2]_j}} \tag{43}$$

Substitution of **(34)** and **(42)** into **(43)**, gives:

$$t_{[\hat{\underline{b}}_2]_j} \Rightarrow \frac{[\underline{\mathbf{b}}_2]_j}{\left(\Sigma_{0,0}\right)^{0.5}\left[\left(\int \underline{b}_2 \underline{b}_2'\right)^{-0.5}\right]_{jj}} \tag{44}$$

This implies that the t-ratios for the $I(1)$ coefficients converge to a random variable, in the limit, and not zero (as one might expect of a regression of independently generated $I(0)$ variables on $I(1)$ processes). Thus, there is potential for spurious significance for these regressors, depending upon the mean value of the random variable that the t-ratios converge to relative to their critical values – see Stewart (2006). Indeed, such potential spurious correlation can give rise to spurious cointegration.

The coefficient of determination is given by:

$$R^2 = \frac{\hat{\underline{b}}_1' \sum_{t=1}^{T} \underline{x}_{1t} \underline{x}_{1t}' \hat{\underline{b}}_1 + 2\hat{\underline{b}}_2' \sum_{t=1}^{T} \underline{x}_{2t} \underline{x}_{1t}' \hat{\underline{b}}_1 + \hat{\underline{b}}_2' \sum_{t=1}^{T} \underline{x}_{2t} \underline{x}_{2t}' \hat{\underline{b}}_2}{\sum_{t=1}^{T} y_{0t}^2} \tag{45}$$

Substitution of **(17)**, **(18)**, **(20)**, **(21)**, **(32)** and **(34)** into **(45)**, gives:

$$TR^2 \Rightarrow \frac{\underline{\mathbf{b}}_1' \underline{\Sigma}_{1,0} \underline{\mathbf{b}}_1 + \underline{\mathbf{b}}_2' \int \underline{b}_2 \underline{b}_2' \underline{\mathbf{b}}_2}{\Sigma_{0,0}} \qquad \textbf{(46)}$$

In the limit the coefficient of determination tends to zero at rate T. This implies that the coefficient of determination does not exhibit spurious regression for this model specification.

Following Hassler (1996), the F-statistics for testing the joint significance of all of the *I*(0) regressors' coefficients ($H_0 : \underline{b}_1 = \underline{0}$), denoted F_1, and the joint significance of all of the *I*(1) variables' coefficients ($H_0 : \underline{b}_2 = \underline{0}$), denoted F_2, are given below.

$$F_1 = \frac{\hat{\underline{b}}_1' A^{-1} \hat{\underline{b}}_1}{s^2 m_1} \qquad \textbf{(47)}$$

Substitution of **(28)**, **(32)** and **(36)** into **(47)**, after taking the probability limit, yields:

$$p \lim F_1 = \frac{\underline{\mathbf{b}}_1' \underline{\Sigma}_{1,0} \underline{\mathbf{b}}_1}{\Sigma_{0,0}\, m_1} \qquad \textbf{(48)}$$

This implies that the F-statistic for the joint significance of all of the *I*(0) regressors' coefficients converges to a random variable, in the probability limit. Therefore, there is potential joint spurious significance for these variables.

$$F_2 = \frac{\hat{\underline{b}}_2' D^{-1} \hat{\underline{b}}_2}{s^2 m_2} \qquad \textbf{(49)}$$

Substitution of **(30)**, **(34)** and **(36)** into **(49)**, yields:

$$F_2 \Rightarrow \frac{\underline{\mathbf{b}}_2' \int \underline{b}_2 \underline{b}_2' \underline{\mathbf{b}}_2}{\Sigma_{0,0}\, m_2} \qquad \textbf{(50)}$$

Equation **(50)** implies that the F-statistic for the joint significance of all of the *I*(1) regressors' coefficients converges to a random variable, and not zero, in the limit. Thus, this statistic features potential spurious regression.

The Durbin-Watson statistic for testing first-order autocorrelation can be expressed as:

$$DW = \frac{T^{-1}\sum_{t=2}^{T}\left[\left(v_{0t} - v_{0t-1}\right) - \underline{\hat{b}}_{1}'\left(\underline{v}_{1t} - \underline{v}_{1t-1}\right) - \underline{\hat{b}}_{2}'\,\underline{v}_{2t}\right]^{2}}{s^{2}} \qquad \textbf{(51)}$$

Substitution of **(11)**, **(12)**, **(22)**, **(32)** and **(34)** into **(51)**, with some additional assumptions (given in the footnote), yields:

$$p\lim DW = 2\left[1 - \left(\frac{\Sigma_{0,1}}{\Sigma_{0,0}}\right)\right] \qquad \textbf{(52)}$$

The Durbin-Watson statistic converges to a constant in the probability limit. Whether it is indicative of autocorrelation depends upon the degree of autocorrelation in the dependent variable. This can be illustrated by assuming that y_{0t} follows a first-order autoregressive scheme, thus:

$$y_{0t} = r y_{0t-1} + e_{0t}, \qquad -1 < r < 1 \qquad \textbf{(53)}$$

where, $\mathrm{E}(e_{0t}) = 0$; $\mathrm{E}(e_{0t}^{2}) = s_{0}^{2}$; $\mathrm{E}(e_{0t}e_{0s}) = 0$, $t \neq s$; with, $s = 1, 2, ..., T$.

The OLS estimator of $\hat{r}$ in **(53)**, in the probability limit, is:[31]

$$p\lim \hat{r} = \frac{\Sigma_{0,1}}{\Sigma_{0,0}} \qquad \textbf{(54)}$$

Substitution of **(54)** into **(52)** gives:

$$p\lim DW = 2\left(1 - \hat{r}\right) \qquad \textbf{(55)}$$

This familiar form of DW suggests that when y_{0t} is not autocorrelated ($\hat{r} = 0$) then DW converges to 2 suggesting no autocorrelation in the residuals of equation **(16)**. However, when the dependent variable is autocorrelated ($\hat{r} \neq 0$), DW may indicate significant autocorrelation in the residuals, $\hat{u}_{1t}$. This result arises because, given that all of the coefficients in regression **(16)** converge to zero in the probability limit (from **(32)** and **(34)**), $\hat{u}_{1t}$ will exhibit exactly the same autocorrelation properties as y_{0t}. Hence the relationship between the degree of autocorrelation in y_{0t} and DW.

From Park and Phillips (1989, Lemma 2.1, p. 99) the following can be obtained:[32]

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[31] That is, the OLS estimator is: $\hat{r} = \frac{\sum_{t=2}^{T} y_{0t-1}y_{0t}}{\sum_{t=2}^{T} y_{0t-1}^{2}} = \frac{T^{-1}\sum_{t=2}^{T} v_{0t-1}v_{0t}}{T^{-1}\sum_{t=2}^{T} v_{0t-1}v_{0t-1}} \Rightarrow p\lim \hat{r} = \frac{\Sigma_{0,1}}{\Sigma_{0,0}}$.

$$T^{0.5}\overline{Y}_0 \Rightarrow B_0(1) \tag{56}$$

$$T^{0.5}\overline{\underline{X}}_1 \Rightarrow \underline{B}_1(1) \tag{57}$$

$$T^{-0.5}\overline{\underline{X}}_2 \Rightarrow \int_0^1 \underline{B}_2(r)dr =: \int \underline{B}_2 \tag{58}$$

The formula for the OLS estimator of the intercept in **(1)** is:

$$\hat{a} = \overline{Y}_0 - \hat{\underline{b}}_1{}' \overline{\underline{X}}_1 - \hat{\underline{b}}_2{}' \overline{\underline{X}}_2 \tag{59}$$

Substitution of **(32)**, **(34)**, **(56)**, **(57)** and **(58)** into **(59)**, gives:

$$T^{0.5}\hat{a} \Rightarrow B_0(1) - \underline{\mathbf{b}}_2{}' \int \underline{B}_2 \tag{60}$$

The OLS estimator for the intercept converges to zero at rate $T^{0.5}$.

In summary, for the regression of an autocorrelated stationary variable on the set of *I*(0) regressors and a set of *I*(1) variables we find the following. The statistics, $\hat{\underline{b}}_1$, $\hat{\underline{b}}_2$ and R^2 do not exhibit spurious correlation. That is, the coefficients on both *I*(0) and *I*(1) regressors and the coefficient of determination all converge to zero in the probability limit. In contrast, there is potential spurious significance for the statistics $t_{\hat{\underline{b}}_1}$, $t_{\hat{\underline{b}}_2}$, F_1 and F_2. In other words, the t-ratios on the *I*(0) and *I*(1) variables, the F-test for the joint significance of the *I*(0) regressors and the F-test for the joint significance of the *I*(1) regressors tend to random variables in the probability limit. Monte Carlo simulations may be employed to determine whether, and the extent to which, these statistics exhibit spurious correlation. It is also evident that the Durbin-Watson statistic will only indicate autocorrelation when the dependent variable is autocorrelated.[33]

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[32] Park and Phillips (1989, Lemma 2.1, p. 99) demonstrate that: $p\lim_{T\to\infty} T^{-0.5}\sum_{t=1}^{T} \underline{X}_{1t} = \underline{B}_1(1)$ and $p\lim T^{-1.5}\sum_{t=1}^{T} \underline{X}_{2t} = \int_0^1 \underline{B}_2(r)dr =: \int \underline{B}_2$. Banerjee *et al* (2003, p. 27) provide the fuller expression of the Brownian motion for a single *I*(1) process.

[33] In addition, as long as the dependent variable is stationary ($\hat{r} < 1$) the DW statistic will virtually always exceed zero asymptotically which implies that there is potential spurious regression (manifested in exaggerated t-ratios and F-statistics). Given that R^2 tends to zero in the probability limit spurious significance occurs when $R^2 < DW$. Thus, whilst $R^2 > DW$ is traditionally indicative of spurious correlation, $R^2 < DW$ does not necessarily imply the absence of nonsense regression.

3.2 Regressions of an I(0) variable on I(1) regressors

We now consider the unbalanced regression of an $I(0)$ dependent variable on a set of m_2 $I(1)$ explanatory factors. The model, in mean deviation form, is essentially equation **(16)** with the constraint $\underline{\hat{b}}_1 = \underline{0}$ imposed. That is:

$$y_{0t} = \underline{\tilde{b}}_2{}' \underline{x}_{2t} + \hat{u}_{2t} \qquad \textbf{(61)}$$

The OLS coefficient estimator is defined as:

$$\underline{\tilde{b}}_2 = \left(\sum_{t=1}^{T} \underline{x}_{2t} \underline{x}_{2t}{}' \right)^{-1} \sum_{t=1}^{T} \underline{x}_{2t} y_{0t} \qquad \textbf{(62)}$$

Substitution of **(18)** and **(19)** into **(62)**, gives:

$$T \underline{\tilde{b}}_2 \Rightarrow \left(\int \underline{b}_2 \underline{b}_2{}' \right)^{-1} \int \underline{b}_2{}' dB_0 = \underline{\mathbf{b}}_2^* \qquad \textbf{(63)}$$

The OLS estimator $\underline{\tilde{b}}_2$ converges to zero at rate T : there is no spurious correlation in terms of the coefficient estimator.

The residual variance is given by:[34]

$$\tilde{s}^2 = T^{-1} \sum_{t=1}^{T} \left(y_{0t} - \underline{\tilde{b}}_2{}' \underline{x}_{2t} \right)^2 \qquad \textbf{(64)}$$

Substitution of **(18)**, **(19)**, **(20)** and **(63)** into **(64)**, after taking the probability limit, yields:

$$p\lim \tilde{s}^2 = \Sigma_{0,0} \qquad \textbf{(65)}$$

The residual variance converges to a constant equal to the variance of the dependent variable. This arises because the coefficient estimator converges to zero, therefore, from **(61)**, $p\lim y_{0t} = \hat{u}_{2t}$.

The estimator of the standard error for the j$^{\text{th}}$ coefficient is:

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[34] This derives from the standard formula: $\tilde{s}^2 = T^{-1} \sum_{t=1}^{T} \hat{u}_{2t}^2$. This is asymptotically equivalent to the unbiased estimator of the regression standard error.

$$s_{[\tilde{\underline{b}}_2]_j} = \left[\tilde{s}^2\left(\sum_{t=1}^{T}\underline{x}_{2t}\underline{x}_{2t}'\right)^{-1}_{jj}\right]^{0.5} \tag{66}$$

Substitution of **(18)** and **(65)** into **(66)**, gives:

$$Ts_{[\tilde{\underline{b}}_2]_j} \Rightarrow \Sigma_{0,0}^{0.5}\left(\int \underline{b}_2\underline{b}_2'\right)^{-0.5}_{jj} \tag{67}$$

The coefficient standard error converges to zero at rate T.

The jth coefficient's t-ratio is:

$$\tilde{t}_j = \frac{\left[\tilde{\underline{b}}_2\right]_j}{s_{[\tilde{\underline{b}}_2]_j}} \tag{68}$$

Substitution of **(63)** and **(67)** into **(68)**, yields:

$$\tilde{t}_j \Rightarrow \frac{\left[\underline{\mathbf{b}}_2^*\right]_j}{\Sigma_{0,0}^{0.5}\left(\int \underline{b}_2\underline{b}_2'\right)^{-0.5}_{jj}} \tag{69}$$

The t-ratio of each of the j $I(1)$ regressors' coefficients converges to a random variable (and not zero) in the limit, thus giving rise to the possibility of spurious correlation in the model. This potential spurious correlation suggests the possibility of spurious cointegration.

The coefficient of determination is given by:

$$R^2 = \frac{\tilde{\underline{b}}_2'\sum_{t=1}^{T}\underline{x}_{2t}\underline{x}_{2t}'\tilde{\underline{b}}_2}{\sum_{t=1}^{T}y_{0t}^2} \tag{70}$$

Substitution of **(18)**, **(20)** and **(63)** into **(70)**, gives:

$$TR^2 \Rightarrow \frac{\underline{\mathbf{b}}_2^{*\prime}\int \underline{b}_2\underline{b}_2'\,\underline{\mathbf{b}}_2^*}{\Sigma_{0,0}} \tag{71}$$

In the limit the coefficient of determination tends to zero at rate T. This statistic does not exhibit spurious regression in this model.

The F-statistic for the joint redundancy of all of the $I(1)$ regressors is defined as:

$$\tilde{F}_2 = \left(\frac{T - m_2 - 1}{m_2}\right)\left(\frac{R^2}{1 - R^2}\right) \tag{72}$$

Substitution of **(71)** into **(72)**, yields:

$$\tilde{F}_2 \Rightarrow \frac{\underline{\mathbf{b}}_2^{*\prime} \int \underline{b}_2 \underline{b}_2^{\prime}\, \underline{\mathbf{b}}_2^{*}}{m_2 \Sigma_{0,0}} \tag{73}$$

This F-statistic converges to a random variable (rather than zero) which suggests that this statistic will potentially exhibit spurious significance.

The Durbin-Watson statistic for testing first-order autocorrelation is:

$$DW = \frac{T^{-1} \sum_{t=2}^{T} \left[\left(v_{0t} - v_{0t-1} \right) - \hat{\underline{b}}_2^{\prime}\, \underline{v}_{2t} \right]^2}{\tilde{s}^2} \tag{74}$$

Substitution of **(11)**, **(12)**, **(63)** and **(65)** into **(74)**, with some additional assumptions (given in the footnote), yields:

$$p \lim DW = 2\left[1 - \left(\frac{\Sigma_{0,1}}{\Sigma_{0,0}}\right)\right] \tag{75}$$

The Durbin Watson statistic can be shown to be related to the degree of autocorrelation in the dependent variable (as measured by $\hat{r}$) in the same way as above (equation **(55)**):

$$p \lim DW = 2(1 - \hat{r}) \tag{76}$$

The degree of autocorrelation in the residuals, as indicated by DW, mirrors the degree of autocorrelation in the dependent variable. This is because the coefficients on the $I(1)$ variables converge to zero in the probability limit (equation **(63)**) and so the autocorrelation properties of $\hat{u}_{2t}$ and y_{0t} are exactly the same – from equation **(61)**.

The formula for the OLS estimator of the intercept in **(1)**, with $\hat{\underline{b}}_1^{\prime} = 0$, is:

$$\hat{a} = \bar{Y}_0 - \underline{\tilde{b}}_2{}' \underline{\bar{X}}_2 \qquad \textbf{(77)}$$

Substitution of **(63)**, **(56)** and **(58)** into **(77)** gives:

$$T^{0.5}\hat{a} \Rightarrow B_0(1) - \underline{\mathbf{b}}_2^{*\prime} \int \underline{B}_2 \qquad \textbf{(78)}$$

The OLS estimator for the intercept converges to zero at rate $T^{0.5}$.

In summary, for the regression of an autocorrelated stationary variable on a set of *I*(1) regressors we find the following. The statistics, $\underline{\tilde{b}}_2$ and R^2 do not exhibit spurious correlation because they converge to zero asymptotically. In contrast, there is potential spurious significance for the statistics $\tilde{t}_{\underline{\tilde{b}}_2}$ and $\tilde{F}_2$. In other words, the t-ratios on the *I*(1) variables and the F-test for the joint significance of the *I*(1) regressors tend to random variables in the probability limit. Monte Carlo simulations can be used to determine whether, and the extent to which, these statistics exhibit spurious correlation. We also find that the Durbin-Watson statistic will only indicate residual autocorrelation when the dependent variable is autocorrelated.[35]

4. Concluding remarks

Using asymptotic theory we have demonstrated that the t-ratios (the F-statistic) for the *I*(1) regressors in a model with an *I*(0) dependent variable (with or without additional *I*(0) regressors) converge to a random variable and not zero. Stewart (2006) argues that such a result can be consistent with spurious rejections if the means of the *I*(1) variables' t-ratios (F-statistic) converge to *large* values (relative to their critical values). Within the context of the two-variable LRM there is some limited Monte Carlo simulation evidence in terms of the rejections of the regressor's t-ratio in a regression of an *I*(0) variable on an *I*(1) process. Banerjee *et al* (1993) find evidence against spurious rejections when the *I*(0) variable and the first difference of the *I*(1) regressor are independently generated white noise processes. However, Granger *et al* (2001) find that there is evident spurious regression when the dependent variable is a first-order autoregressive process with autocorrelation coefficient equal to 0.5 and the *I*(1) regressor's first-difference is white noise. Thus, there is some limited simulation evidence which demonstrates spurious significance of the *I*(1) regressors' t-ratios in a regression with an *I*(0) dependent variable. However, no evidence is available for the case when an *I*(0) explanatory factor is added to this model. Providing (further) simulation evidence for both of these models is an area for further work (which is underway).

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[35] In addition, provided that the dependent variable is stationary we find that, asymptotically, $R^2 < DW$ at the same time as $\tilde{t}_{\underline{\tilde{b}}_2}$ and $\tilde{F}_2$ exhibit spurious significance. Hence, $R^2 < DW$ does not necessarily imply the absence of nonsense regression in some form.

It is also found that the error term in the models considered here will reflect, for example, the autocorrelation properties of the dependent variable. Thus, autocorrelation in the dependent variable will translate into residual autocorrelation that will in turn affect the distributions of t- and F-statistics on the $I(1)$ terms (as well as any $I(0)$ regressors). This would explain the finding of spurious regression and suggests that a means of ameliorating it is to model the dynamics present in the dependent variable so that the residuals are white noise. Alternatively, heteroscedasticity and autocorrelation consistent standard errors may be employed to reduce or remove this spurious significance.

Given that the specifications examined in this paper are of the form of an ECM under the null of no cointegration, a finding of spurious rejections for the $I(1)$ terms would suggest that there is potential for inferring spurious cointegration from such models.[36] Although there is no Monte Carlo simulation evidence for the t-ratio in this particular form of ECM the results of Franses (1990) suggest that spurious cointegration may be found. Further work producing simulation evidence on the existence, or otherwise, of spurious cointegration in such an ECM is warranted (and is underway).[37]

Thus, this paper suggests that spurious correlation can be found in models where previously it was not thought be present. It also suggests that spurious correlation in such a specification can give rise to spurious cointegration.

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[36] Note that in an error-correction model the assumption that all variables are generated completely independently needs to be relaxed given that the $I(0)$ variables are typically the first differences of the $I(1)$ terms.

[37] This will require the use of the critical values provided by Ericcson and MacKinnon (2002).

A1. Appendix 1: Derivation of $p\lim\left(T^{d}\sum_{t=1}^{T} v_{0t}\underline{v}_{1t}\right)$

Let the single process (variable), y_{0t}, be defined as:

$$Y_{0t} = v_{0t} \tag{A1.1}$$

Let the $m_1 \times 1$ vector of processes, $\underline{X}_{1t}{}' = \left[X_{1t} \quad X_{2t} \quad \ldots \quad X_{m_1 t}\right]$, be defined as:

$$\underline{X}_{1t} = \underline{v}_{1t} \tag{A1.2}$$

where $\underline{v}_{1t}{}' = \left[v_{1t} \quad v_{2t} \quad \ldots \quad v_{m_1 t}\right]$. A bar below a symbol denotes a vector or matrix and t = 1, 2,…, T.

It is assumed that y_{0t} and $\underline{x}_{1t}$ are zero mean, independently generated stationary processes that are homoscedastic and feature first-order autocorrelation, thus:

First-order autoregressions:

$$v_{0t} = r v_{0t-1} + e_{0t}, \qquad -1 < r < 1 \tag{A1.3}$$

where, $\mathrm{E}(e_{0t}) = 0$; $\mathrm{E}(e_{0t}^2) = s_0^2$; $\mathrm{E}(e_{0t}e_{0s}) = 0$, $\mathrm{t} \neq s$; $\mathrm{E}(e_{0t}v_{0s})$, $\mathrm{t} > s$, where, $\mathrm{s} = 1, 2, \ldots, \mathrm{T}$.

$$v_{it} = f_i v_{it-1} + e_{it}, \qquad -1 < f_i < 1, \qquad \mathrm{i} = 1,2,\ldots m_1. \tag{A1.4}$$

where, $\mathrm{E}(e_{it}) = 0$; $\mathrm{E}(e_{it}^2) = s_i^2$; $\mathrm{E}(e_{it}e_{is}) = 0$, $\mathrm{t} \neq s$; $\mathrm{E}(e_{it}v_{is})$, $\mathrm{t} > s$.

Zero mean:[38]

$$E(v_{0t}) = E(\underline{v}_{1t}) = E(v_{it}) = 0, \qquad \mathrm{i} = 0,1,2,\ldots,m_1 \tag{A1.5}$$

Homoscedastic:[39]

$$Var(v_{0t}) = E(v_{0t}^2) = s_0^2 = \left(\frac{1}{1-r^2}\right)s_{e_0}^2 = \Sigma_{0,0} \tag{A1.6}$$

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[38] For example, $E(v_{0t}) = r^{T}E(v_{0t-T}) + \sum_{q=0}^{T-1} r^{q}E(e_{0t-q}) = 0$ if either the initial value $e_{0t-q} = 0$ or if T tends to infinity.

[39] See, for example, Pindyck and Rubinfeld (1997, p. 528).

$$Var(v_{it}) = E(v_{it}^2) = s_i^2 = \left(\frac{1}{1-f_i^2}\right) s_{e_i}^2, \qquad \text{i} = 1,2,...m_1. \qquad \textbf{(A1.7)}$$

$$Var(\underline{v}_{1t}) = E(\underline{v}_{1t}\underline{v}_{1t}{}') = \underline{\Sigma}_{1,0} = \begin{bmatrix} E(v_{1t}^2) & E(v_{1t}v_{2t}) & \dots & E(v_{1t}v_{m_1t}) \\ E(v_{2t}v_{1t}) & E(v_{2t}^2) & \dots & E(v_{2t}v_{m_1t}) \\ \dots & \dots & \dots & \dots \\ E(v_{m_1t}v_{1t}) & E(v_{m_1t}v_{2t}) & \dots & E(v_{m_1t}^2) \end{bmatrix} \qquad \textbf{(A1.8)}$$

Equation **(A1.8)** will be simplified below.

Autocorrelated:

$$E(v_{0t}v_{0s}) = r^{|t-s|} s_0^2 = \left(\frac{r^{|t-s|}}{1-r^2}\right) s_{e_0}^2, \qquad t \neq s \qquad \textbf{(A1.9)}$$

$$E(v_{it}v_{is}) = f_i^{|t-s|} s_i^2 = \left(\frac{f_i^{|t-s|}}{1-f_i^2}\right) s_{e_i}^2, \qquad t \neq s, \qquad \text{i} = 1,2,...,m_1. \qquad \textbf{(A1.10)}$$

Independent:[40]

$$E(v_{0t} \mid \underline{v}_{1s}) = E(\underline{v}_{1s} \mid v_{0t}) = E(v_{it} \mid v_{js}) = 0, \qquad \text{i} \neq \text{j}. \qquad \textbf{(A1.11)}$$

The assumption of independence can be used to show that the off-the-principal-diagonal elements of **(A1.8)** are equal to zero as follows. From the Law of Iterated Expectations it is known that:[41]

$$E(v_{it}v_{js}) = E_{v_{js}}[E(v_{it}v_{js} \mid v_{js})] \qquad \textbf{(A1.12)}$$

where $E_{v_{js}}(\bullet)$ is the expectation taken over the values of v_{js}.

Because v_{js} is being conditioned upon on the right-hand side of **(A1.12)** it can be taken outside of the expectation, thus:

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[40] Equation **(A1.11)** is the result of the assumption of independence and the zero mean of errors assumption – see equation **(A1.5)**. Independence means that the occurrence or non-occurrence of v_{it} does not affect the occurrence or non-occurrence of v_{js} and vice-versa. Note that when $s = t$ independence refers to contemporaneous observations of different processes and when $s \neq t$ it refers to non-contemporaneous observations.

[41] The Law of Iterated Expectations can be expressed, for two random variables X and Y, as: $E(XY) = E_X[E(Y \mid X)]$, see Greene (2003, p. 865).

$$E(v_{it}v_{js}) = E_{v_{js}}[E(v_{it}v_{js} \mid v_{js})] = E_{v_{js}}[v_{js}E(v_{it} \mid v_{js})] \qquad \textbf{(A1.13)}$$

Invoking the assumption of independence, equation **(A1.11)**, in equation **(A1.13)** gives:

$$E(v_{it}v_{js}) = E_{v_{js}}[v_{js} \times 0] = 0, \qquad i \neq j \qquad \textbf{(A1.14)}$$

Substitution of equation **(A1.14)**, with $s = t$, *and* equation **(A1.7)** into equation **(A1.8)** yields:

$$\textbf{(A1.15)}$$

$$Var(\underline{v}_{1t}) = E(\underline{v}_{1t}\underline{v}_{1t}') = \underline{\Sigma}_{1,0} = \begin{bmatrix} s_1^2 & 0 & \dots & 0 \\ 0 & s_2^2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & s_{m_1}^2 \end{bmatrix} = \begin{bmatrix} s_1^2 & s_2^2 & \dots & s_{m_1}^2 \end{bmatrix} \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix} = \underline{s}_1^{2\,\prime} \underline{I}_{m_1}$$

where $\underline{I}_{m_1}$ is the $m_1 \times m_1$ identity matrix and $\underline{s}_1^{2\,\prime} = \begin{bmatrix} s_1^2 & s_2^2 & \dots & s_{m_1}^2 \end{bmatrix}$.

To determine $p\lim\left(T^d \sum_{t=1}^{T} v_{0t}\underline{v}_{1t}\right)$ it is necessary to consider what $T^d \sum_{t=1}^{T} v_{0t}\underline{v}_{1t}$ converges to in quadratic mean.[42] In particular, this involves finding both the mean and variance of $T^d \sum_{t=1}^{T} v_{0t}\underline{v}_{1t}$ in the limit (as T tends to infinity) – see Greene (2003, p. 897). Both are shown in turn.

The limit of the mean of $T^d \sum_{t=1}^{T} v_{0t}\underline{v}_{1t}$

The mean of $T^d \sum_{t=1}^{T} v_{0t}\underline{v}_{1t}$ is:

$$\textbf{(A1.16)}$$

$$E\left(T^d \sum_{t=1}^{T} v_{0t}\underline{v}_{1t}\right) = T^d E(v_{01}\underline{v}_{11} + v_{02}\underline{v}_{12} + \dots + v_{0T}\underline{v}_{1T}) = T^d[E(v_{01}\underline{v}_{11}) + E(v_{02}\underline{v}_{12}) + \dots + E(v_{0T}\underline{v}_{1T})]$$

$$\Rightarrow E\left(T^d \sum_{t=1}^{T} v_{0t}\underline{v}_{1t}\right) = T^d \sum_{t=1}^{T} E(v_{0t}\underline{v}_{1t}) \qquad \textbf{(A1.17)}$$

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[42] Convergence in quadratic mean is a special case of convergence in probability.

By the Law of Iterated Expectations (see equation **(A1.12)** and **(A1.13)** above):[43]

$$E(v_{0t}\underline{v}_{1t}) = E_{\underline{v}_{1t}}[E(v_{0t}\underline{v}_{1t} \mid \underline{v}_{1t})] = E_{\underline{v}_{1t}}[\underline{v}_{1t}E(v_{0t} \mid \underline{v}_{1t})] \qquad \textbf{(A1.18)}$$

where $E_{\underline{v}_{1t}}(\bullet)$ is the expectation taken over the values of $\underline{v}_{1t}$.

Substituting the assumption of independence, equation **(A1.11)**, into equation **(A1.18)** gives:

$$E(v_{0t}\underline{v}_{1t}) = E_{\underline{v}_{1t}}[0 \times E(v_{0t} \mid \underline{v}_{1t})] = \underline{0} \qquad \textbf{(A1.19)}$$

Substitution of equation **(A1.19)** in to **(A1.17)** yields:

$$E\left(T^{d}\sum_{t=1}^{T} v_{0t}\underline{v}_{1t}\right) = T^{d}\sum_{t=1}^{T}(\underline{0}) = \underline{0} \qquad \textbf{(A1.20)}$$

Since the exact expectation of $T^{d}\sum_{t=1}^{T} v_{0t}\underline{v}_{1t}$ is zero (that is, for all sample sizes – values of T), the limit of $T^{d}\sum_{t=1}^{T} v_{0t}\underline{v}_{1t}$ (as T tends to infinity) is also zero. That is:

$$\underset{T\to\infty}{Lim}\ E\left(T^{d}\sum_{t=1}^{T} v_{0t}\underline{v}_{1t}\right) = \underline{0} \qquad \textbf{(A1.21)}$$

Equation **(A1.21)** establishes that **the mean of $T^{d}\sum_{t=1}^{T} v_{0t}\underline{v}_{1t}$ converges to the constant value zero in the limit (as T tends to infinity)**. **This is true for ALL values of d .**

Next the value of d for which the variance of $T^{d}\sum_{t=1}^{T} v_{0t}\underline{v}_{1t}$ converges to a matrix containing finite non-zero values in the limit is derived.

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[43] Because $\underline{v}_{1t}$ is being conditioned upon it is taken outside of the expectation on the right-hand side of equation **(A1.18)**. This can equivalently be derived as follows. Given that all terms in the following covariance are zero mean (see equation **(A1.5)**) it is clear that: $E(v_{0t}\underline{v}_{1t}) = Cov(v_{0t}\underline{v}_{1t})$. Applying the following rule for covariances, $Cov(XY) = Cov_{X}[X, E(Y \mid X)]$, (see, Greene, 2003, p. 865), yields: $E(v_{0t}\underline{v}_{1t}) = Cov(v_{0t}\underline{v}_{1t}) = Cov_{\underline{v}_{1t}}[\underline{v}_{1t}E(v_{0t} \mid \underline{v}_{1t})]$. Given that all processes in the covariance are zero mean implies that: $Cov_{\underline{v}_{1t}}[\underline{v}_{1t}E(v_{0t} \mid \underline{v}_{1t})] = E_{\underline{v}_{1t}}[\underline{v}_{1t}E(v_{0t} \mid \underline{v}_{1t})]$. Hence, $E(v_{0t}\underline{v}_{1t}) = E_{\underline{v}_{1t}}[\underline{v}_{1t}E(v_{0t} \mid \underline{v}_{1t})]$.

The limit of the variance of $T^{d}\sum_{t=1}^{T} v_{0t}\underline{v}_{1t}$

Defining the following:

$$\underline{w}_{t} = v_{0t}\underline{v}_{1t} \qquad \textbf{(A1.22)}$$

Means that:

$$\overline{\underline{w}} = T^{d}\sum_{t=1}^{T}\underline{w}_{t} = T^{d}\sum_{t=1}^{T} v_{0t}\underline{v}_{1t} = T^{d}\underline{V}_{1}{'}\,\underline{v}_{0} \qquad \textbf{(A1.23)}$$

where, $\underline{v}_{0}$ is a $T \times 1$ vector and $\underline{V}_{1}$ a $T \times m_{1}$ matrix that are defined as follows:

$$\underline{v}_{0} = \begin{bmatrix} v_{01} \\ v_{02} \\ \cdots \\ v_{0T} \end{bmatrix} \qquad \textbf{(A1.24)}$$

$$\underline{V}_{1} = \begin{bmatrix} v_{11} & v_{21} & \cdots & v_{m_{1}1} \\ v_{12} & v_{22} & \cdots & v_{m_{1}2} \\ \cdots & \cdots & \cdots & \cdots \\ v_{1T} & v_{2T} & \cdots & v_{m_{1}T} \end{bmatrix} \qquad \textbf{(A1.25)}$$

Taking the variance of **(A1.23)** gives:

$$Var\left(\overline{\underline{w}}\right) = Var\left(T^{d}\sum_{t=1}^{T} v_{0t}\underline{v}_{1t}\right) \qquad \textbf{(A1.26)}$$

Applying the Decomposition of Variance rule to **(A1.26)** gives:[44]

$$Var\left(\overline{\underline{w}}\right) = Var_{\underline{v}_{1t}}\left[E\left(\overline{\underline{w}} \mid \underline{v}_{1t}\right)\right] + E_{\underline{v}_{1t}}\left[Var\left(\overline{\underline{w}} \mid \underline{v}_{1t}\right)\right] \qquad \textbf{(A1.27)}$$

First, consider the $E\left(\overline{\underline{w}} \mid \underline{v}_{1t}\right)$ term in equation **(A1.27)** using **(A1.23)**, thus:

$$E\left(\overline{\underline{w}} \mid \underline{v}_{1t}\right) = E\left[\left(T^{d}\underline{V}_{1}{'}\,\underline{v}_{0}\right) \mid \underline{V}_{1}\right] = T^{d}E\left[\left(\underline{V}_{1}{'}\,\underline{v}_{0}\right) \mid \underline{V}_{1}\right] \qquad \textbf{(A1.28)}$$

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[44] The Decomposition of Variance rule is, $Var(Y) = Var_{X}[E(Y \mid X)] + E_{X}[Var(Y \mid X)]$, where $E_{X}(\bullet)$ and $Var_{X}(\bullet)$ are the expectation and variance taken over the values of X – see Greene, 2003, p. 866.

Because the expectation in **(A1.28)** conditions on $\underline{V}_1$, this means that $\underline{V}_1$ can be taken outside of the expectation operator, thus:

$$E\left(\overline{\underline{w}} \mid \underline{v}_{1t}\right) = T^d \underline{V}_1{}' E\left(\underline{v}_0 \mid \underline{V}_1\right) \qquad \textbf{(A1.29)}$$

Invoking the assumption of independence, equation **(A1.11)**, $E(v_{0t} \mid \underline{v}_{1t}) = 0 \Rightarrow E\left(\underline{v}_0 \mid \underline{V}_1\right) = \underline{0}$, in equation **(A1.29)**, gives:[45]

$$E\left(\overline{\underline{w}} \mid \underline{v}_{1t}\right) = T^d \underline{V}_1{}' \underline{0} = \underline{0} \qquad \textbf{(A1.30)}$$

Substitution of **(A1.30)** into **(A1.27)** yields:

$$Var\left(\overline{\underline{w}}\right) = Var_{\underline{v}_{1t}}\left(\underline{0}\right) + E_{\underline{v}_{1t}}\left[Var\left(\overline{\underline{w}} \mid \underline{v}_{1t}\right)\right] = E_{\underline{v}_{1t}}\left[Var\left(\overline{\underline{w}} \mid \underline{v}_{1t}\right)\right] \qquad \textbf{(A1.31)}$$

Given that all processes are zero mean, equation **(A1.5)**, the term $Var\left(\overline{\underline{w}} \mid \underline{v}_{1t}\right)$ in equation **(A1.31)** can be written as:

$$Var\left(\overline{\underline{w}} \mid \underline{v}_{1t}\right) = E\left(\overline{\underline{w}}\,\overline{\underline{w}}{}' \mid \underline{v}_{1t}\right) \qquad \textbf{(A1.32)}$$

Substitution of equation **(A1.23)**, $\overline{\underline{w}} = T^d \underline{V}_1{}' \underline{v}_0$, into **(A1.32)** gives:

$$Var\left(\overline{\underline{w}} \mid \underline{v}_{1t}\right) = E\left[\left(T^d \underline{V}_1{}' \underline{v}_0\right)\left(T^d \underline{V}_1{}' \underline{v}_0\right)' \mid \underline{V}_1\right] = E\left(T^{2d} \underline{V}_1{}' \underline{v}_0 \underline{v}_0{}' \underline{V}_1 \mid \underline{V}_1\right) \qquad \textbf{(A1.33)}$$

Because the expectation in equation **(A1.33)** conditions upon $\underline{V}_1$, this term can be treated as a constant and be taken outside of the expectation operator, thus:

$$Var\left(\overline{\underline{w}} \mid \underline{v}_{1t}\right) = T^{2d} \underline{V}_1{}' \left[E\left(\underline{v}_0 \underline{v}_0{}'\right)\right] \underline{V}_1 \qquad \textbf{(A1.34)}$$

The term $E\left(\underline{v}_0 \underline{v}_0{}'\right)$ can be expressed in detail as:

(A1.35)

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[45] Where $\underline{0}$ is the null vector/matrix.

$$E\left(\underline{v}_0\underline{v}_0'\right)=\begin{bmatrix} E\left(v_{01}^2\right) & E\left(v_{01}v_{02}\right) & \dots & E\left(v_{01}v_{0T}\right) \\ E\left(v_{02}v_{01}\right) & E\left(v_{02}^2\right) & \dots & E\left(v_{02}v_{0T}\right) \\ \dots & \dots & \dots & \dots \\ E\left(v_{0T}v_{01}\right) & E\left(v_{0T}v_{02}\right) & \dots & E\left(v_{0T}^2\right) \end{bmatrix}=\begin{bmatrix} E\left(v_{01}^2\right) & E\left(v_{0t}v_{0t-1}\right) & \dots & E\left(v_{0t}v_{0t-(T-1)}\right) \\ E\left(v_{0t}v_{0t-1}\right) & E\left(v_{02}^2\right) & \dots & E\left(v_{0t}v_{0t-(T-2)}\right) \\ \dots & \dots & \dots & \dots \\ E\left(v_{0t}v_{0t-(T-1)}\right) & E\left(v_{0t}v_{0t-(T-2)}\right) & \dots & E\left(v_{0T}^2\right) \end{bmatrix}$$

Substitution of equation **(A1.6)**, $E(v_{0t}^2)=s_0^2=\left(\frac{1}{1-r^2}\right)s_{e_0}^2$, and **(A1.9)**, $E(v_{0t}v_{0t-k})=r^k s_0^2=\left(\frac{r^k}{1-r^2}\right)s_{e_0}^2$, into **(A1.35)** gives:

$$E\left(\underline{v}_0\underline{v}_0'\right)=\begin{bmatrix} s_0^2 & r s_0^2 & \dots & r^{T-1}s_0^2 \\ r s_0^2 & s_0^2 & \dots & r^{T-2}s_0^2 \\ \dots & \dots & \dots & \dots \\ r^{T-1}s_0^2 & r^{T-2}s_0^2 & \dots & s_0^2 \end{bmatrix}=\left(\frac{1}{1-r^2}\right)s_{e_0}^2\begin{bmatrix} 1 & r & \dots & r^{T-1} \\ r & 1 & \dots & r^{T-2} \\ \dots & \dots & \dots & \dots \\ r^{T-1} & r^{T-2} & \dots & 1 \end{bmatrix}=\underline{\Lambda}_0\left(\frac{1}{1-r^2}\right)s_{e_0}^2 \quad \textbf{(A1.36)}$$

where,

$$\underline{\Lambda}_0=\begin{bmatrix} 1 & r & \dots & r^{T-1} \\ r & 1 & \dots & r^{T-2} \\ \dots & \dots & \dots & \dots \\ r^{T-1} & r^{T-2} & \dots & 1 \end{bmatrix} \quad \textbf{(A1.37)}$$

Substitution of **(A1.36)** into **(A1.34)**, $Var\left(\underline{\overline{w}}\,|\,\underline{v}_{1t}\right)=T^{2d}\underline{V}_1'\left[E\left(\underline{v}_0\underline{v}_0'\right)\right]\underline{V}_1$, gives:

$$Var\left(\underline{\overline{w}}\,|\,\underline{v}_{1t}\right)=T^{2d}s_{e_0}^2\left(\frac{1}{1-r^2}\right)\underline{V}_1'\,\underline{\Lambda}_0\underline{V}_1 \quad \textbf{(A1.38)}$$

Substitution of **(A1.38)** into **(A1.31)**, $Var\left(\underline{\overline{w}}\right)=E_{\underline{v}_{1t}}\left[Var\left(\underline{\overline{w}}\,|\,\underline{v}_{1t}\right)\right]$, yields:

$$Var\left(\underline{\overline{w}}\right)=E_{\underline{v}_{1t}}\left[T^{2d}s_{e_0}^2\left(\frac{1}{1-r^2}\right)\underline{V}_1'\,\underline{\Lambda}_0\underline{V}_1\right]=T^{2d}s_{e_0}^2\left(\frac{1}{1-r^2}\right)E_{\underline{v}_{1t}}\left(\underline{V}_1'\,\underline{\Lambda}_0\underline{V}_1\right) \quad \textbf{(A1.39)}$$

Writing the matrices of the term $\underline{V}_1'\,\underline{\Lambda}_0\underline{V}_1$ in detail, using **(A1.25)** and **(A1.37)**, gives:

$$\underline{V}_1{}' \, \underline{\Lambda}_0 \underline{V}_1 = \begin{bmatrix} v_{11} & v_{12} & \dots & v_{1T} \\ v_{21} & v_{22} & \dots & v_{2T} \\ \dots & \dots & \dots & \dots \\ v_{m_1 1} & v_{m_1 2} & \dots & v_{m_1 T} \end{bmatrix} \begin{bmatrix} 1 & r & \dots & r^{T-1} \\ r & 1 & \dots & r^{T-2} \\ \dots & \dots & \dots & \dots \\ r^{T-1} & r^{T-2} & \dots & 1 \end{bmatrix} \begin{bmatrix} v_{11} & v_{21} & \dots & v_{m_1 1} \\ v_{12} & v_{22} & \dots & v_{m_1 2} \\ \dots & \dots & \dots & \dots \\ v_{1T} & v_{2T} & \dots & v_{m_1 T} \end{bmatrix} \quad \textbf{(A1.40)}$$

$$\Rightarrow \underline{V}_1{}' \underline{\Lambda}_0 \underline{V}_1 = \begin{bmatrix} (v_{11} + rv_{12} + \ldots + r^{T-1} v_{1T}) & (rv_{11} + v_{12} + \ldots + r^{T-2} v_{1T}) & \ldots & (r^{T-1} v_{11} + r^{T-2} v_{12} + \ldots + v_{1T}) \\ (v_{21} + rv_{22} + \ldots + r^{T-1} v_{2T}) & (rv_{21} + v_{22} + \ldots + r^{T-2} v_{2T}) & \ldots & (r^{T-1} v_{21} + r^{T-2} v_{22} + \ldots + v_{2T}) \\ \ldots & \ldots & \ldots & \ldots \\ (v_{m_1 1} + rv_{m_1 2} + \ldots + r^{T-1} v_{m_1 T}) & (rv_{m_1 1} + v_{m_1 2} + \ldots + r^{T-2} v_{m_1 T}) & \ldots & (r^{T-1} v_{m_1 1} + r^{T-2} v_{m_1 2} + \ldots + v_{m_1 T}) \end{bmatrix} \begin{bmatrix} v_{11} & v_{21} & \ldots & v_{m_1 1} \\ v_{12} & v_{22} & \ldots & v_{m_1 2} \\ \ldots & \ldots & \ldots & \ldots \\ v_{1T} & v_{2T} & \ldots & v_{m_1 T} \end{bmatrix} \qquad \textbf{(A1.41)}$$

$$\Rightarrow \underline{V}_1{}' \underline{\Lambda}_0 \underline{V}_1 = \begin{bmatrix} \sum_{t=1}^{T} \sum_{s=1}^{T} r^{|t-s|} v_{1t} v_{1s} & \sum_{t=1}^{T} \sum_{s=1}^{T} r^{|t-s|} v_{1t} v_{2s} & \ldots & \sum_{t=1}^{T} \sum_{s=1}^{T} r^{|t-s|} v_{1t} v_{m_1 s} \\ \sum_{t=1}^{T} \sum_{s=1}^{T} r^{|t-s|} v_{2t} v_{1s} & \sum_{t=1}^{T} \sum_{s=1}^{T} r^{|t-s|} v_{2t} v_{2s} & \ldots & \sum_{t=1}^{T} \sum_{s=1}^{T} r^{|t-s|} v_{2t} v_{m_1 s} \\ \ldots & \ldots & \ldots & \ldots \\ \sum_{t=1}^{T} \sum_{s=1}^{T} r^{|t-s|} v_{m_1 t} v_{1s} & \sum_{t=1}^{T} \sum_{s=1}^{T} r^{|t-s|} v_{m_1 t} v_{2s} & \ldots & \sum_{t=1}^{T} \sum_{s=1}^{T} r^{|t-s|} v_{m_1 t} v_{m_1 s} \end{bmatrix} \qquad \textbf{(A1.42)}$$

where, $\sum_{t=1}^{T} \sum_{s=1}^{T} r^{|t-s|} v_{it} v_{js} = (v_{i1} v_{j1} + r v_{i2} v_{j1} + \ldots + r^{T-1} v_{iT} v_{j1}) + (r v_{i1} v_{j2} + v_{i2} v_{j2} + \ldots + r^{T-2} v_{iT} v_{j2}) + \ldots + (r^{T-1} v_{i1} v_{jT} + r^{T-2} v_{i2} v_{jT} + \ldots + v_{iT} v_{jT})$.[46]

Substitution of **(A1.42)** into **(A1.39)** gives:

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[46] For example, when $i = j = 1$, this implies that: $\sum_{t=1}^{T} \sum_{s=1}^{T} r^{|t-s|} v_{1t} v_{1s} = (v_{11}^2 + r v_{12} v_{11} + \ldots + r^{T-1} v_{1T} v_{11}) + (r v_{11} v_{12} + v_{12}^2 + \ldots + r^{T-2} v_{1T} v_{12}) + \ldots + (r^{T-1} v_{11} v_{1T} + r^{T-2} v_{12} v_{1T} + \ldots + v_{1T}^2)$ and, when $i = 1$ and $j = 2$: $\sum_{t=1}^{T} \sum_{s=1}^{T} r^{|t-s|} v_{1t} v_{2s} = (v_{11} v_{21} + r v_{12} v_{21} + \ldots + r^{T-1} v_{1T} v_{21}) + (r v_{11} v_{22} + v_{12} v_{22} + \ldots + r^{T-2} v_{1T} v_{22}) + \ldots + (r^{T-1} v_{11} v_{2T} + r^{T-2} v_{12} v_{2T} + \ldots + v_{1T} v_{2T})$, and so on.

$$Var\left(\underline{\overline{w}}\right)=T^{2d}s_{e_0}^{2}\left(\frac{1}{1-r^{2}}\right)\begin{bmatrix}\sum_{t=1}^{T}\sum_{s=1}^{T}r^{|t-s|}E_{\underline{v}_{1t}}\left(v_{1t}v_{1s}\right) & \sum_{t=1}^{T}\sum_{s=1}^{T}r^{|t-s|}E_{\underline{v}_{1t}}\left(v_{1t}v_{2s}\right) & \ldots & \sum_{t=1}^{T}\sum_{s=1}^{T}r^{|t-s|}E_{\underline{v}_{1t}}\left(v_{1t}v_{m_1s}\right)\\ \sum_{t=1}^{T}\sum_{s=1}^{T}r^{|t-s|}E_{\underline{v}_{1t}}\left(v_{2t}v_{1s}\right) & \sum_{t=1}^{T}\sum_{s=1}^{T}r^{|t-s|}E_{\underline{v}_{1t}}\left(v_{2t}v_{2s}\right) & \ldots & \sum_{t=1}^{T}\sum_{s=1}^{T}r^{|t-s|}E_{\underline{v}_{1t}}\left(v_{2t}v_{m_1s}\right)\\ \ldots & \ldots & \ldots & \ldots\\ \sum_{t=1}^{T}\sum_{s=1}^{T}r^{|t-s|}E_{\underline{v}_{1t}}\left(v_{m_1t}v_{1s}\right) & \sum_{t=1}^{T}\sum_{s=1}^{T}r^{|t-s|}E_{\underline{v}_{1t}}\left(v_{m_1t}v_{2s}\right) & \ldots & \sum_{t=1}^{T}\sum_{s=1}^{T}r^{|t-s|}E_{\underline{v}_{1t}}\left(v_{m_1t}v_{m_1s}\right)\end{bmatrix} \qquad \textbf{(A1.43)}$$

For each element of the matrix in equation **(A1.43)** the assumptions of homoscedasticity, $E(v_{it}^{2})=\left(\frac{1}{1-f_i^{2}}\right)s_{e_i}^{2}$, **(A1.7)**; autocorrelation, $E(v_{it}v_{is})=\left(\frac{f_i^{|t-s|}}{1-f_i^{2}}\right)s_{e_i}^{2}$, **(A1.10)**; and (non-) contemporaneous independence, $E(v_{it}v_{js})=0$, **(A1.14)**, can be invoked as follows:[47]

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[47] Notice that when $i=j$ in equation **(A1.44)** then the expectation operator is applied to products of the same error process in either the same time period (where homoscedasticity is assumed) or different time periods (where the autocorrelation properties are invoked). However, when $i\neq j$ in equation **(A1.44)** the expectation operator is applied to the product of different error processes, either in the same time period (contemporaneous) or different time periods (non-contemporaneous), and so the independence assumption is applied. For example, for the first two elements in equation **(A1.43)**:

$$\sum_{t=1}^{T}\sum_{s=1}^{T}r^{|t-s|}E_{\underline{v}_{1t}}\left(v_{1t}v_{1s}\right)=\left[E_{\underline{v}_{1t}}\left(v_{11}^{2}\right)+rE_{\underline{v}_{1t}}\left(v_{12}v_{11}\right)+\ldots+r^{T-1}E_{\underline{v}_{1t}}\left(v_{1T}v_{11}\right)\right]+\left[rE_{\underline{v}_{1t}}\left(v_{11}v_{12}\right)+E_{\underline{v}_{1t}}\left(v_{12}^{2}\right)+\ldots+r^{T-2}E_{\underline{v}_{1t}}\left(v_{1T}v_{12}\right)\right]+\ldots+\left[r^{T-1}E_{\underline{v}_{1t}}\left(v_{11}v_{1T}\right)+r^{T-2}E_{\underline{v}_{1t}}\left(v_{12}v_{1T}\right)+\ldots+E_{\underline{v}_{1t}}\left(v_{1T}^{2}\right)\right]$$

$$\Rightarrow\sum_{t=1}^{T}\sum_{s=1}^{T}r^{|t-s|}E_{\underline{v}_{1t}}\left(v_{1t}v_{1s}\right)=\left\{\left[\left(\frac{1}{1-f_1^{2}}\right)s_{e_1}^{2}\right]+\left[r\left(\frac{f_1}{1-f_1^{2}}\right)s_{e_1}^{2}\right]+\ldots+\left[r^{T-1}\left(\frac{f_1^{T-1}}{1-f_1^{2}}\right)s_{e_1}^{2}\right]\right\}+\left\{\left[r\left(\frac{f_1}{1-f_1^{2}}\right)s_{e_1}^{2}\right]+\left[\left(\frac{1}{1-f_1^{2}}\right)s_{e_1}^{2}\right]+\ldots+\left[r^{T-2}\left(\frac{f_1^{T-2}}{1-f_1^{2}}\right)s_{e_1}^{2}\right]\right\}+\ldots+\left\{\left[r^{T-1}\left(\frac{f_1^{T-1}}{1-f_1^{2}}\right)s_{e_1}^{2}\right]+\left[r^{T-2}\left(\frac{f_1^{T-2}}{1-f_1^{2}}\right)s_{e_1}^{2}\right]+\ldots+\left[\left(\frac{1}{1-f_1^{2}}\right)s_{e_1}^{2}\right]\right\}$$

$$\sum_{t=1}^{T}\sum_{s=1}^{T}r^{|t-s|}E_{\underline{v}_{1t}}\left(v_{1t}v_{2s}\right)=\left[E_{\underline{v}_{1t}}\left(v_{11}v_{21}\right)+rE_{\underline{v}_{1t}}\left(v_{12}v_{21}\right)+\ldots+r^{T-1}E_{\underline{v}_{1t}}\left(v_{1T}v_{21}\right)\right]+\left[rE_{\underline{v}_{1t}}\left(v_{11}v_{22}\right)+E_{\underline{v}_{1t}}\left(v_{12}v_{22}\right)+\ldots+r^{T-2}E_{\underline{v}_{1t}}\left(v_{1T}v_{22}\right)\right]+\ldots+\left[r^{T-1}E_{\underline{v}_{1t}}\left(v_{11}v_{2T}\right)+r^{T-2}E_{\underline{v}_{1t}}\left(v_{12}v_{2T}\right)+\ldots+E_{\underline{v}_{1t}}\left(v_{1T}v_{2T}\right)\right]$$

$$\Rightarrow\sum_{t=1}^{T}\sum_{s=1}^{T}r^{|t-s|}E_{\underline{v}_{1t}}\left(v_{1t}v_{2s}\right)=\left[0+\left(r\times0\right)+\ldots+\left(r^{T-1}\times0\right)\right]+\left[\left(r\times0\right)+0+\ldots+\left(r^{T-2}\times0\right)\right]+\ldots+\left[\left(r^{T-1}\times0\right)+\left(r^{T-2}\times0\right)+\ldots+0\right]=0$$

(A1.44)

$$\sum_{t=1}^{T}\sum_{s=1}^{T} r^{|t-s|} E_{\underline{v}_{1t}}\left(v_{it}v_{js}\right)=\begin{cases}\left\{\left[\left(\frac{1}{1-f_i^2}\right)s_{e_i}^2\right]+\left[r\left(\frac{f_i}{1-f_i^2}\right)s_{e_i}^2\right]+\dots+\left[r^{T-1}\left(\frac{f_i^{T-1}}{1-f_i^2}\right)s_{e_i}^2\right]\right\}+\left\{\left[r\left(\frac{f_i}{1-f_i^2}\right)s_{e_i}^2\right]+\left[\left(\frac{1}{1-f_i^2}\right)s_{e_i}^2\right]+\dots+\left[r^{T-2}\left(\frac{f_i^{T-2}}{1-f_i^2}\right)s_{e_i}^2\right]\right\} \\ +\dots+\left\{\left[r^{T-1}\left(\frac{f_i^{T-1}}{1-f_i^2}\right)s_{e_i}^2\right]+\left[r^{T-2}\left(\frac{f_i^{T-2}}{1-f_i^2}\right)s_{e_i}^2\right]+\dots+\left[\left(\frac{1}{1-f_i^2}\right)s_{e_i}^2\right]\right\}, \quad i=j \\ \\ 0, \quad i\neq j\end{cases}$$

Substitution of **(A1.44)** into **(A1.43)** yields:

(A1.45)

$$Var\left(\overline{\underline{\omega}}\right)=T^{2d}\sigma_{\varepsilon_0}^{2}\left(\frac{1}{1-\rho^{2}}\right)\begin{bmatrix}
\begin{array}{l}
\left\{\left[\left(\frac{1}{1-\phi_1^2}\right)\sigma_{\varepsilon_1}^2\right]+\ldots+\left[\rho^{T-1}\left(\frac{\phi_1^{T-1}}{1-\phi_1^2}\right)\sigma_{\varepsilon_1}^2\right]\right\}\\
+\left\{\left[\rho\left(\frac{\phi_1}{1-\phi_1^2}\right)\sigma_{\varepsilon_1}^2\right]+\ldots+\left[\rho^{T-2}\left(\frac{\phi_1^{T-2}}{1-\phi_1^2}\right)\sigma_{\varepsilon_1}^2\right]\right\}\\
+\ldots+\left\{\left[\rho^{T-1}\left(\frac{\phi_1^{T-1}}{1-\phi_1^2}\right)\sigma_{\varepsilon_1}^2\right]+\ldots+\left[\left(\frac{1}{1-\phi_1^2}\right)\sigma_{\varepsilon_1}^2\right]\right\}
\end{array} & 0 & \ldots & 0\\
0 & \begin{array}{l}
\left\{\left[\left(\frac{1}{1-\phi_2^2}\right)\sigma_{\varepsilon_2}^2\right]+\ldots+\left[\rho^{T-1}\left(\frac{\phi_2^{T-1}}{1-\phi_2^2}\right)\sigma_{\varepsilon_2}^2\right]\right\}\\
+\left\{\left[\rho\left(\frac{\phi_2}{1-\phi_2^2}\right)\sigma_{\varepsilon_2}^2\right]+\ldots+\left[\rho^{T-2}\left(\frac{\phi_2^{T-2}}{1-\phi_2^2}\right)\sigma_{\varepsilon_2}^2\right]\right\}\\
+\ldots+\left\{\left[\rho^{T-1}\left(\frac{\phi_2^{T-1}}{1-\phi_2^2}\right)\sigma_{\varepsilon_2}^2\right]+\ldots+\left[\left(\frac{1}{1-\phi_2^2}\right)\sigma_{\varepsilon_2}^2\right]\right\}
\end{array} & \ldots & 0\\
\ldots & \ldots & \ldots & \ldots\\
0 & 0 & \ldots & \begin{array}{l}
\left\{\left[\left(\frac{1}{1-\phi_{m_1}^2}\right)\sigma_{\varepsilon_{m1}}^2\right]+\ldots+\left[\rho^{T-1}\left(\frac{\phi_{m_1}^{T-1}}{1-\phi_{m_1}^2}\right)\sigma_{\varepsilon_{m1}}^2\right]\right\}\\
+\left\{\left[\rho\left(\frac{\phi_{m_1}}{1-\phi_{m_1}^2}\right)\sigma_{\varepsilon_{m1}}^2\right]+\ldots+\left[\rho^{T-2}\left(\frac{\phi_{m_1}^{T-2}}{1-\phi_{m_1}^2}\right)\sigma_{\varepsilon_{m1}}^2\right]\right\}\\
+\ldots+\left\{\left[\rho^{T-1}\left(\frac{\phi_{m_1}^{T-1}}{1-\phi_{m_1}^2}\right)\sigma_{\varepsilon_{m1}}^2\right]+\ldots+\left[\left(\frac{1}{1-\phi_{m_1}^2}\right)\sigma_{\varepsilon_{m1}}^2\right]\right\}
\end{array}
\end{bmatrix}$$

(A1.46)

$$Var\left(\underline{\overline{w}}\right)=T^{2d}s_{e_0}^{2}\left(\frac{1}{1-r^{2}}\right)\begin{bmatrix} \left[\left(\frac{1}{1-f_1^{2}}\right)s_{e_1}^{2}\right]\begin{bmatrix}\left(1+rf_1+\ldots+r^{T-1}f_1^{T-1}\right)\\ +\left(rf_1+1+\ldots+r^{T-2}f_1^{T-2}\right)+\ldots\\ +\left(r^{T-1}f_1^{T-1}+r^{T-2}f_1^{T-2}+\ldots+1\right)\end{bmatrix} & 0 & \ldots & 0\\ 0 & \left[\left(\frac{1}{1-f_2^{2}}\right)s_{e_2}^{2}\right]\begin{bmatrix}\left(1+rf_2+\ldots+r^{T-1}f_2^{T-1}\right)\\ +\left(rf_2+1+\ldots+r^{T-2}f_2^{T-2}\right)+\ldots\\ +\left(r^{T-1}f_2^{T-1}+r^{T-2}f_2^{T-2}+\ldots+1\right)\end{bmatrix} & \ldots & 0\\ \ldots & \ldots & \ldots & \ldots\\ 0 & 0 & \ldots & \left[\left(\frac{1}{1-f_{m_1}^{2}}\right)s_{e_{m1}}^{2}\right]\begin{bmatrix}\left(1+rf_{m_1}+\ldots+r^{T-1}f_{m_1}^{T-1}\right)\\ +\left(rf_{m_1}+1+\ldots+r^{T-2}f_{m_1}^{T-2}\right)+\ldots\\ +\left(r^{T-1}f_{m_1}^{T-1}+r^{T-2}f_{m_1}^{T-2}+\ldots+1\right)\end{bmatrix}\end{bmatrix}$$

The second term of the product of the elements along the principal diagonal of the matrix in equation **(A1.46)** may be expressed, in general, as:

(A1.47)

$$\begin{aligned}&\left(1+rf_i+\ldots+r^{T-1}f_i^{T-1}\right)+\left(rf_i+1+rf_i+\ldots+r^{T-2}f_i^{T-2}\right)+\ldots+\left(r^{T-1}f_i^{T-1}+r^{T-2}f_i^{T-2}+\ldots+1\right)\\ &=\left(1+rf_i+\ldots+r^{T-1}f_i^{T-1}\right)+\ldots+\left(r^{T-n}f_i^{T-n}+\ldots+1+\ldots+r^{T-n}f_i^{T-n}+r^{T-n+1}f_i^{T-n+1}\right)+\left(r^{T-n+1}f_i^{T-n+1}+\ldots+1+\ldots+r^{T-n}f_i^{T-n}\right)+\ldots+\left(r^{T-1}f_i^{T-1}+r^{T-2}f_i^{T-2}+\ldots+1\right)\end{aligned}$$

where, $n=\frac{T}{2}$, assuming, for simplicity, that T is an even number, and $i=1,2,\ldots,m_1$.[48]

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[48] When T is an odd number and $n=\frac{T+1}{2}$, equation **(A1.48)** can be re-written as:

$$\begin{aligned}&\left(1+rf_i+\ldots+r^{T-1}f_i^{T-1}\right)+\left(rf_i+1+rf_i+\ldots+r^{T-2}f_i^{T-2}\right)+\ldots+\left(r^{T-1}f_i^{T-1}+r^{T-2}f_i^{T-2}+\ldots+1\right)=+\left(r^{T-n}f_i^{T-n}+r^{T-n}f_i^{T-n}+\ldots+1+\ldots+r^{T-n}f_i^{T-n}+r^{T-n}f_i^{T-n}\right)\\ &\qquad+2\left[\left(1+rf_i+\ldots+r^{T-1}f_i^{T-1}\right)+\left(rf_i+1+rf_i+\ldots+r^{T-2}f_i^{T-2}\right)+\ldots+\left(r^{T-n-1}f_i^{T-n-1}+r^{T-n}f_i^{T-n}+\ldots+1+\ldots+r^{T-n}f_i^{T-n}+r^{T-n+1}f_i^{T-n+1}\right)\right]\end{aligned}$$

(A1.48a)

$$\begin{aligned}&\Rightarrow \left(1+rf_i+\ldots+r^{T-1}f_i^{T-1}\right)+\left(rf_i+1+rf_i+\ldots+r^{T-2}f_i^{T-2}\right)+\ldots+\left(r^{T-1}f_i^{T-1}+r^{T-2}f_i^{T-2}+\ldots+1\right)\\ &\quad=2\left[\left(1+rf_i+\ldots+r^{T-1}f_i^{T-1}\right)+\left(rf_i+1+rf_i+\ldots+r^{T-2}f_i^{T-2}\right)+\ldots+\left(r^{T-n}f_i^{T-n}+r^{T-n}f_i^{T-n}+\ldots+1+\ldots+r^{T-n}f_i^{T-n}+r^{T-n+1}f_i^{T-n+1}\right)\right]\end{aligned} \quad \textbf{(A1.48)}$$

Equation **(A1.48)** can be simplified using geometric progressions as follows:[49] [50]

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[49] The geometric progressions used to obtain equation **(A1.49)** from equation **(A1.48)** are as follows:
Let, $S_1=\left(1+rf_i+\ldots+r^{T-1}f_i^{T-1}\right)$, $\Rightarrow rf_iS_1=\left(rf_i+r^2f_i^2+\ldots+r^Tf_i^T\right)$, $\Rightarrow\left(S_1-rf_iS_1\right)=\left(1+rf_i+r^2f_i^2+\ldots+r^{T-1}f_i^{T-1}\right)-\left(rf_i+r^2f_i^2+\ldots+r^{T-1}f_i^{T-1}+r^Tf_i^T\right)$,
$\Rightarrow\left(1-rf_i\right)S_1=\left(1-r^Tf_i^T\right)$, $\therefore S_1=\left(1+rf_i+\ldots+r^{T-1}f_i^{T-1}\right)=\dfrac{\left(1-r^Tf_i^T\right)}{\left(1-rf_i\right)}$. Let, $S_2=\left(rf_i+1+rf_i+\ldots+r^{T-2}f_i^{T-2}\right)$, $\Rightarrow rf_iS_2=\left(r^2f_i^2+rf_i+r^2f_i^2+\ldots+r^{T-1}f_i^{T-1}\right)$,
$\Rightarrow\left(S_2-rf_iS_2\right)=\left(rf_i+1+rf_i+r^2f_i^2+\ldots+r^{T-2}f_i^{T-2}\right)-\left(r^2f_i^2+rf_i+r^2f_i^2+\ldots+r^{T-2}f_i^{T-2}+r^{T-1}f_i^{T-1}\right)$, $\Rightarrow\left(1-rf_i\right)S_2=\left(1+rf_i-r^2f_i^2-r^{T-1}f_i^{T-1}\right)$,
$\therefore S_2=\left(rf_i+1+rf_i+\ldots+r^{T-2}f_i^{T-2}\right)=\dfrac{\left(1+rf_i-r^2f_i^2-r^{T-1}f_i^{T-1}\right)}{\left(1-rf_i\right)}$. Let, $S_3=\left(r^2f_i^2+rf_i+1+rf_i+\ldots+r^{T-3}f_i^{T-3}\right)$,
$\Rightarrow rf_iS_3=\left(r^3f_i^3+r^2f_i^2+rf_i+r^2f_i^2+\ldots+r^{T-2}f_i^{T-2}\right)$,
$\Rightarrow\left(S_3-rf_iS_3\right)=\left(r^2f_i^2+rf_i+1+rf_i+r^2f_i^2+\ldots+r^{T-3}f_i^{T-3}\right)-\left(r^3f_i^3+r^2f_i^2+rf_i+r^2f_i^2+\ldots+r^{T-3}f_i^{T-3}+r^{T-2}f_i^{T-2}\right)$, $\Rightarrow\left(1-rf_i\right)S_3=\left(1+rf_i-r^3f_i^3-r^{T-2}f_i^{T-2}\right)$,
$\therefore S_3=\left(r^2f_i^2+rf_i+1+rf_i+\ldots+r^{T-3}f_i^{T-3}\right)=\dfrac{\left(1+rf_i-r^3f_i^3-r^{T-2}f_i^{T-2}\right)}{\left(1-rf_i\right)}$. Let, $S_n=\left(r^{T-n+1}f_i^{T-n+1}+r^{T-n}f_i^{T-n}+\ldots+1+\ldots+r^{T-n}f_i^{T-n}\right)$,
$\Rightarrow rf_iS_n=\left(r^{T-n+2}f_i^{T-n+2}+r^{T-n+1}f_i^{T-n+1}+\ldots+rf_i+\ldots+r^{T-n}f_i^{T-n}+r^{T-n+1}f_i^{T-n+1}\right)$,
$\Rightarrow\left(S_n-rf_iS_n\right)=\left(r^{T-n}f_i^{T-n}+\ldots+rf_i+1+rf_i\ldots+r^{T-n}f_i^{T-n}+r^{T-n+1}f_i^{T-n+1}\right)-\left(r^{T-n+1}f_i^{T-n+1}+r^{T-n}f_i^{T-n}+\ldots+rf_i+\ldots+r^{T-n+1}f_i^{T-n+1}+r^{T-n+2}f_i^{T-n+2}\right)$,
$\Rightarrow\left(1-rf_i\right)S_n=\left(1+rf_i-r^{T-n+1}f_i^{T-n+1}-r^{T-n+2}f_i^{T-n+2}\right)$, $\therefore S_n=\left(r^{T-n+1}f_i^{T-n+1}+r^{T-n}f_i^{T-n}+\ldots+1+\ldots+r^{T-n}f_i^{T-n}\right)=\dfrac{\left(1+rf_i-r^{T-n+1}f_i^{T-n+1}-r^{T-n+2}f_i^{T-n+2}\right)}{\left(1-rf_i\right)}$.

[50] When T is an odd number and $n=\dfrac{T+1}{2}$, equation **(A1.49)** can be re-written as:

$$\left(1+rf_i+\ldots+r^{T-1}f_i^{T-1}\right)+\left(rf_i+1+rf_i+\ldots+r^{T-2}f_i^{T-2}\right)+\ldots+\left(r^{T-1}f_i^{T-1}+r^{T-2}f_i^{T-2}+\ldots+1\right)$$

$$=2\left\{\left[\frac{\left(1-r^{T}f_i^{T}\right)}{\left(1-rf_i\right)}\right]+\left[\frac{\left(1+rf_i-r^{2}f_i^{2}-r^{T-1}f_i^{T-1}\right)}{\left(1-rf_i\right)}\right]+\left[\frac{\left(1+rf_i-r^{3}f_i^{3}-r^{T-2}f_i^{T-2}\right)}{\left(1-rf_i\right)}\right]+\ldots+\left[\frac{\left(1+rf_i-r^{T-n+1}f_i^{T-n+1}-r^{T-n+2}f_i^{T-n+2}\right)}{\left(1-rf_i\right)}\right]\right\} \quad \textbf{(A1.49)}$$

(A1.50)

$$\Rightarrow \left(1+rf_i+\ldots+r^{T-1}f_i^{T-1}\right)+\left(rf_i+1+rf_i+\ldots+r^{T-2}f_i^{T-2}\right)+\ldots+\left(r^{T-1}f_i^{T-1}+r^{T-2}f_i^{T-2}+\ldots+1\right)=\frac{T+(T-2)rf_i-2r^{2}f_i^{2}-2r^{3}f_i^{3}-\ldots-r^{T-1}f_i^{T-1}-r^{T}f_i^{T}}{\left(1-rf_i\right)}$$

$$\left(1+rf_i+\ldots+r^{T-1}f_i^{T-1}\right)+\left(rf_i+1+rf_i+\ldots+r^{T-2}f_i^{T-2}\right)+\ldots+\left(r^{T-1}f_i^{T-1}+r^{T-2}f_i^{T-2}+\ldots+1\right)$$

$$=2\left\{\left[\frac{\left(1-r^{T}f_i^{T}\right)}{\left(1-rf_i\right)}\right]+\left[\frac{\left(1+rf_i-r^{2}f_i^{2}-r^{T-1}f_i^{T-1}\right)}{\left(1-rf_i\right)}\right]+\left[\frac{\left(1+rf_i-r^{3}f_i^{3}-r^{T-2}f_i^{T-2}\right)}{\left(1-rf_i\right)}\right]+\ldots+\left[\frac{\left(1+rf_i-r^{T-n}f_i^{T-n}-r^{T-n+2}f_i^{T-n+2}\right)}{\left(1-rf_i\right)}\right]\right\} \quad \textbf{(A1.49a)}$$

$$+\left[\frac{\left(1+rf_i-2r^{T-n+1}f_i^{T-n+1}\right)}{\left(1-rf_i\right)}\right]$$

This uses the following additional geometric progressions:

Let, $S_{n_1}=\left(r^{T-n-1}f_i^{T-n-1}+\ldots+rf_i+1+rf_i+\ldots+r^{T-n}f_i^{T-n}+r^{T-n+1}f_i^{T-n+1}\right)$, $\Rightarrow rf_iS_{n_1}=\left(r^{T-n}f_i^{T-n}+r^{T-n-1}f_i^{T-n-1}+\ldots+rf_i+\ldots+r^{T-n+1}f_i^{T-n+1}+r^{T-n+2}f_i^{T-n+2}\right)$,

$\Rightarrow \left(S_{n_1}-rf_iS_{n_1}\right)=\left(r^{T-n-1}f_i^{T-n-1}+\ldots+rf_i+1+rf_i+\ldots+r^{T-n}f_i^{T-n}+r^{T-n+1}f_i^{T-n+1}\right)-\left(r^{T-n}f_i^{T-n}+r^{T-n-1}f_i^{T-n-1}+\ldots+rf_i+\ldots+r^{T-n+1}f_i^{T-n+1}+r^{T-n+2}f_i^{T-n+2}\right)$,

$\Rightarrow \left(1-rf_i\right)S_{n_1}=\left(1+rf_i-r^{T-n}f_i^{T-n}-r^{T-n+2}f_i^{T-n+2}\right)$, $\therefore S_{n_1}=\left(r^{T-n-1}f_i^{T-n-1}+\ldots+rf_i+1+rf_i+\ldots+r^{T-n}f_i^{T-n}+r^{T-n+1}f_i^{T-n+1}\right)=\dfrac{\left(1+rf_i-r^{T-n}f_i^{T-n}-r^{T-n+2}f_i^{T-n+2}\right)}{\left(1-rf_i\right)}$.

Let, $S_{n_2}=\left(r^{T-n}f_i^{T-n}+\ldots+rf_i+1+rf_i+\ldots+r^{T-n}f_i^{T-n}\right)$, $\Rightarrow rf_iS_{n_2}=\left(r^{T-n+1}f_i^{T-n+1}+r^{T-n}f_i^{T-n}+\ldots+rf_i+\ldots+r^{T-n}f_i^{T-n}+r^{T-n+1}f_i^{T-n+1}\right)$,

$\Rightarrow \left(S_{n_2}-rf_iS_{n_2}\right)=\left(r^{T-n}f_i^{T-n}+\ldots+rf_i+1+rf_i+\ldots+r^{T-n}f_i^{T-n}\right)-\left(r^{T-n+1}f_i^{T-n+1}+r^{T-n}f_i^{T-n}+\ldots+rf_i+\ldots+r^{T-n}f_i^{T-n}+r^{T-n+1}f_i^{T-n+1}\right)$,

$\Rightarrow \left(1-rf_i\right)S_{n_2}=\left(1+rf_i-2r^{T-n+1}f_i^{T-n+1}\right)$, $\therefore S_{n_2}=\left(r^{T-n}f_i^{T-n}+\ldots+rf_i+1+rf_i+\ldots+r^{T-n}f_i^{T-n}\right)=\dfrac{\left(1+rf_i-2r^{T-n+1}f_i^{T-n+1}\right)}{\left(1-rf_i\right)}$.

(A1.51)

$$\Rightarrow \left(1+rf_i+\ldots+r^{T-1}f_i^{T-1}\right)+\left(rf_i+1+rf_i+\ldots+r^{T-2}f_i^{T-2}\right)+\ldots+\left(r^{T-1}f_i^{T-1}+r^{T-2}f_i^{T-2}+\ldots+1\right)=\frac{(T+2)+Trf_i-2\left(1+rf_i+r^2f_i^2+r^3f_i^3+\ldots+r^{T-1}f_i^{T-1}+r^Tf_i^T\right)}{(1-rf_i)}$$

Equation **(A1.51)** can be simplified using a geometric progression, thus:[51] [52]

$$\Rightarrow \left(1+rf_i+\ldots+r^{T-1}f_i^{T-1}\right)+\left(rf_i+1+rf_i+\ldots+r^{T-2}f_i^{T-2}\right)+\ldots+\left(r^{T-1}f_i^{T-1}+r^{T-2}f_i^{T-2}+\ldots+1\right)=\frac{(T+2)+Trf_i-2\left[\frac{\left(1-r^{T+1}f_i^{T+1}\right)}{(1-rf_i)}\right]}{(1-rf_i)}$$

(A1.52)

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[51] The geometric progression used is:

Let, $S_{1T}=\left(1+rf_i+\ldots+r^Tf_i^T\right)$, $\Rightarrow rf_iS_{1T}=\left(rf_i+r^2f_i^2+\ldots+r^{T+1}f_i^{T+1}\right)$, $\Rightarrow \left(S_{1T}-rf_iS_{1T}\right)=\left(1+rf_i+r^2f_i^2+\ldots+r^Tf_i^T\right)-\left(rf_i+r^2f_i^2+\ldots+r^{T+1}f_i^{T+1}\right)$,

$\Rightarrow (1-rf_i)S_{1T}=\left(1-r^{T+1}f_i^{T+1}\right)$, $\therefore S_{1T}=\left(1+rf_i+\ldots+r^Tf_i^T\right)=\frac{\left(1-r^{T+1}f_i^{T+1}\right)}{(1-rf_i)}$.

[52] Equation **(A1.49a)**, for when T is an odd number, can be simplified as (which is almost identical to equation **(A1.50)**):

$$\left(1+rf_i+\ldots+r^{T-1}f_i^{T-1}\right)+\left(rf_i+1+rf_i+\ldots+r^{T-2}f_i^{T-2}\right)+\ldots+\left(r^{T-1}f_i^{T-1}+r^{T-2}f_i^{T-2}+\ldots+1\right)=\frac{T+(T-1)rf_i-2r^2f_i^2-2r^3f_i^3-\ldots-r^{T-1}f_i^{T-1}-r^Tf_i^T}{(1-rf_i)}$$

(A1.50a)

$$\Rightarrow \left(1+rf_i+\ldots+r^{T-1}f_i^{T-1}\right)+\left(rf_i+1+rf_i+\ldots+r^{T-2}f_i^{T-2}\right)+\ldots+\left(r^{T-1}f_i^{T-1}+r^{T-2}f_i^{T-2}+\ldots+1\right)=\frac{(T+2)+(T+1)rf_i-2\left(1+rf_i+r^2f_i^2+r^3f_i^3+\ldots+r^{T-1}f_i^{T-1}+r^Tf_i^T\right)}{(1-rf_i)}$$

(A1.51a)

$$\Rightarrow \left(1+rf_i+\ldots+r^{T-1}f_i^{T-1}\right)+\left(rf_i+1+rf_i+\ldots+r^{T-2}f_i^{T-2}\right)+\ldots+\left(r^{T-1}f_i^{T-1}+r^{T-2}f_i^{T-2}+\ldots+1\right)=\frac{(T+2)+(T+1)rf_i-2\left[\frac{\left(1-r^{T+1}f_i^{T+1}\right)}{(1-rf_i)}\right]}{(1-rf_i)}$$

(A1.52a)

$$\Rightarrow \left(1+rf_i+\ldots+r^{T-1}f_i^{T-1}\right)+\left(rf_i+1+rf_i+\ldots+r^{T-2}f_i^{T-2}\right)+\ldots+\left(r^{T-1}f_i^{T-1}+r^{T-2}f_i^{T-2}+\ldots+1\right)=\frac{(1-rf_i)\left[T(1+rf_i)+(2+rf_i)\right]-2\left(1-r^{T+1}f_i^{T+1}\right)}{(1-rf_i)^2}$$

(A1.53a)

This is almost identical to equation **(A1.53)**.

$$\Rightarrow\left(1+rf_i+\ldots+r^{T-1}f_i^{T-1}\right)+\left(rf_i+1+rf_i+\ldots+r^{T-2}f_i^{T-2}\right)+\ldots+\left(r^{T-1}f_i^{T-1}+r^{T-2}f_i^{T-2}+\ldots+1\right)=\frac{(1-rf_i)\left[T(1+rf_i)+2\right]-2\left(1-r^{T+1}f_i^{T+1}\right)}{(1-rf_i)^2} \quad \textbf{(A1.53)}$$

Substitution of **(A1.53)** into **(A1.46)** gives:[53]

$$Var\left(\overline{\underline{w}}\right)=s_{e_0}^2\left(\frac{1}{1-r^2}\right)\begin{bmatrix} \left[\left(\frac{1}{1-f_1^2}\right)s_{e_1}^2\right]\left[\frac{(1-rf_1)\left[T^{2d+1}(1+rf_1)+2T^{2d}\right]-2T^{2d}\left(1-r^{T+1}f_1^{T+1}\right)}{(1-rf_1)^2}\right] & 0 & \ldots & 0 \\ 0 & \left[\left(\frac{1}{1-f_2^2}\right)s_{e_2}^2\right]\left[\frac{(1-rf_2)\left[T^{2d+1}(1+rf_2)+2T^{2d}\right]-2T^{2d}\left(1-r^{T+1}f_2^{T+1}\right)}{(1-rf_2)^2}\right] & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \left[\left(\frac{1}{1-f_{m_1}^2}\right)s_{e_{m1}}^2\right]\left[\frac{(1-rf_{m_1})\left[T^{2d+1}(1+rf_{m_1})+2T^{2d}\right]-2T^{2d}\left(1-r^{T+1}f_{m_1}^{T+1}\right)}{(1-rf_{m_1})^2}\right] \end{bmatrix} \quad \textbf{(A1.54)}$$

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[53] When T is an odd number equation **(A1.54)** becomes:

$$Var\left(\overline{\underline{w}}\right)=s_{e_0}^2\left(\frac{1}{1-r^2}\right)\begin{bmatrix} \left[\left(\frac{1}{1-f_1^2}\right)s_{e_1}^2\right]\left[\frac{(1-rf_1)\left[T^{2d+1}(1+rf_1)+T^{2d}(2+rf_i)\right]-2T^{2d}\left(1-r^{T+1}f_1^{T+1}\right)}{(1-rf_1)^2}\right] & 0 & \ldots & 0 \\ 0 & \left[\left(\frac{1}{1-f_2^2}\right)s_{e_2}^2\right]\left[\frac{(1-rf_2)\left[T^{2d+1}(1+rf_2)+T^{2d}(2+rf_i)\right]-2T^{2d}\left(1-r^{T+1}f_2^{T+1}\right)}{(1-rf_2)^2}\right] & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \left[\left(\frac{1}{1-f_{m_1}^2}\right)s_{e_{m1}}^2\right]\left[\frac{(1-rf_{m_1})\left[T^{2d+1}(1+rf_{m_1})+2T^{2d}\right]-T^{2d}(2+rf_i)\left(1-r^{T+1}f_{m_1}^{T+1}\right)}{(1-rf_{m_1})^2}\right] \end{bmatrix} \quad \textbf{(A1.54a)}$$

In the limit this converges to the same matrix as **(A1.54)**, the case when T is an even number, when $d=-{}^1\!/_2$.

For stationary processes, that is, when $| r | < 1$ and $| f_i | < 1$ that are homoscedastic, every term in equation **(A1.54)** is finite except for T (in the limit), hence:

(1) When $d < -\frac{1}{2}$ the variance of $\left(T^d \sum_{t=1}^{T} v_{0t} \underline{v}_{1t} \right)$ tends to the null matrix in the limit:

$$\underset{T \to \infty}{Lim} \;\; Var\left(\underline{\overline{w}}\right) = \underset{T \to \infty}{Lim} \;\; Var\left(T^d \sum_{t=1}^{T} v_{0t} \underline{v}_{1t} \right) = \begin{bmatrix} 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 \end{bmatrix} = \underline{0}, \qquad d < -\frac{1}{2} \tag{A1.55}$$

Equation **(A1.21)** and equation **(A1.55)** establish that, when $d < -\frac{1}{2}$, the mean and variance of $T^d \sum_{t=1}^{T} v_{0t} \underline{v}_{1t}$ converge to zero in the limit (as T tends to infinity). Hence, when $d < -\frac{1}{2}$, $T^d \sum_{t=1}^{T} v_{0t} \underline{v}_{1t}$ converges to zero in quadratic mean, or the probability limit, thus:

$$p\lim\left(T^d \sum_{t=1}^{T} v_{0t} \underline{v}_{1t} \right) = \underline{0}, \qquad d < -\frac{1}{2} \tag{A1.56}$$

(2) When $d = -\frac{1}{2}$ the variance of $\left(T^{d}\sum_{t=1}^{T} v_{0t}\underline{v}_{1t}\right)$ is:

(A1.57)

$$Var\left(\overline{\underline{w}}\right) = s_{e_0}^{2}\left(\frac{1}{1-r^{2}}\right)\left[\begin{matrix} \left[\left(\frac{1}{1-f_1^{2}}\right)s_{e_1}^{2}\right]\left[\frac{(1-rf_1)\left[(1+rf_1)+2T^{-1}\right]-2T^{-1}\left(1-r^{T+1}f_1^{T+1}\right)}{(1-rf_1)^{2}}\right] & 0 & \dots & 0 \\ 0 & \left[\left(\frac{1}{1-f_2^{2}}\right)s_{e_2}^{2}\right]\left[\frac{(1-rf_2)\left[T(1+rf_2)+2T^{-1}\right]-2T^{-1}\left(1-r^{T+1}f_2^{T+1}\right)}{(1-rf_2)^{2}}\right] & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \left[\left(\frac{1}{1-f_{m_1}^{2}}\right)s_{e_{m1}}^{2}\right]\left[\frac{(1-rf_{m_1})\left[(1+rf_{m_1})+2T^{-1}\right]-2T^{-1}\left(1-r^{T+1}f_{m_1}^{T+1}\right)}{(1-rf_{m_1})^{2}}\right] \end{matrix}\right]$$

Which, in the limit, becomes:[54]

(A1.58)

$$\underset{T\to\infty}{Lim}\ Var\left(\overline{\underline{w}}\right) = \underset{T\to\infty}{Lim}\ Var\left(T^{-\frac{1}{2}}\sum_{t=1}^{T} v_{0t}\underline{v}_{1t}\right) = s_{e_0}^{2}\left(\frac{1}{1-r^{2}}\right)\left[\begin{matrix} \left[\left(\frac{1}{1-f_1^{2}}\right)s_{e_1}^{2}\right]\left[\frac{1-r^{2}f_1^{2}}{(1-rf_1)^{2}}\right] & 0 & \dots & 0 \\ 0 & \left[\left(\frac{1}{1-f_2^{2}}\right)s_{e_2}^{2}\right]\left[\frac{1-r^{2}f_2^{2}}{(1-rf_2)^{2}}\right] & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \left[\left(\frac{1}{1-f_{m_1}^{2}}\right)s_{e_{m1}}^{2}\right]\left[\frac{1-r^{2}f_{m_1}^{2}}{(1-rf_{m_1})^{2}}\right] \end{matrix}\right]$$

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[54] This is true when T is an even number and when T is an odd number.

$$\therefore \underset{T\to\infty}{Lim}\ \ Var\left(T^{-\frac{1}{2}}\sum_{t=1}^{T} v_{0t}\underline{v}_{1t}\right) = s_{e_0}^{2}\left(\frac{1}{1-r^{2}}\right)\underline{\Omega} \qquad \mathbf{(A1.59)}$$

$$\text{where, } \underline{\Omega} = \begin{bmatrix} \left[\left(\frac{1}{1-f_1^{2}}\right)s_{e_1}^{2}\right]\left[\frac{1-r^{2}f_1^{2}}{(1-rf_1)^{2}}\right] & 0 & \ldots & 0 \\ 0 & \left[\left(\frac{1}{1-f_2^{2}}\right)s_{e_2}^{2}\right]\left[\frac{1-r^{2}f_2^{2}}{(1-rf_2)^{2}}\right] & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \left[\left(\frac{1}{1-f_{m_1}^{2}}\right)s_{e_{m1}}^{2}\right]\left[\frac{1-r^{2}f_{m_1}^{2}}{(1-rf_{m_1})^{2}}\right] \end{bmatrix} \qquad \mathbf{(A1.60)}$$

$$\text{and, } \underset{T\to\infty}{Lim}\ \ Var\left(T^{-\frac{1}{2}}\sum_{t=1}^{T} v_{0t}\underline{v}_{1t}\right)_{ii} = s_{e_0}^{2}s_{e_i}^{2}\left[\frac{1-r^{2}f_i^{2}}{(1-r^{2})(1-f_i^{2})(1-rf_i)^{2}}\right], \qquad i = 1, 2, \ldots, m_1 \qquad \mathbf{(A1.61)}$$

All of the elements along the principal diagonal of **(A1.59)** are finite and all of the elements off of the principal diagonal are zero – note that $\underline{\Omega}$ is an $m_1 \times m_1$ matrix and $Var\left(T^{-\frac{1}{2}}\sum_{t=1}^{T} v_{0t}\underline{v}_{1t}\right)_{ii}$ is the element in the ith row and ith column of the $Var\left(T^{-\frac{1}{2}}\sum_{t=1}^{T} v_{0t}\underline{v}_{1t}\right)$ matrix. Hence, in the limit (as T tends to infinity) $Var\left(T^{-\frac{1}{2}}\sum_{t=1}^{T} v_{0t}\underline{v}_{1t}\right)$ is finite. Combining this with equation **(A1.21)**, which establishes that the mean of $T^{-\frac{1}{2}}\sum_{t=1}^{T} v_{0t}\underline{v}_{1t}$ converges to zero in the limit (as T tends to infinity), implies that, $T^{-\frac{1}{2}}\sum_{t=1}^{T} v_{0t}\underline{v}_{1t}$ converges to a random variable, denoted $\underline{\Gamma}$, in quadratic mean, or the probability limit, thus:

$$p\lim T^{-\frac{1}{2}} \sum_{t=1}^{T} v_{0t} \underline{v}_{1t} = \underline{\Gamma} \tag{A1.62}$$

Note that $-\frac{1}{2}$ is the largest value of d where the probability limit converges to a random variable (has a non-zero variance) rather than zero. Hence, equation **(A1.62)** is of interest for the derivation of the probability limit of OLS statistics for the regression of an $I(0)$ dependent variable on a set of $I(0)$ processes and a set of $I(1)$ series.

(3) When $d > -\frac{1}{2}$ the variance of $\left(T^{d} \sum_{t=1}^{T} v_{0t} \underline{v}_{1t} \right)$ tends to the following in the limit:

$$\underset{T \to \infty}{Lim} \; Var\left(\underline{\overline{w}}\right) = \underset{T \to \infty}{Lim} \; Var\left(T^{d} \sum_{t=1}^{T} v_{0t} \underline{v}_{1t} \right) = \begin{bmatrix} \infty & 0 & \ldots & 0 \\ 0 & \infty & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \infty \end{bmatrix} = \underline{\Omega}_{\infty}, \quad d > -\frac{1}{2} \tag{A1.63}$$

Equation **(A1.63)** establishes that, when $d > -\frac{1}{2}$, the elements along the principal diagonal of the variance of $T^{d}\sum_{t=1}^{T} v_{0t} v_{-1t}$ diverge to infinity in the limit (as T tends to infinity). Thus, $T^{d}\sum_{t=1}^{T} v_{0t} v_{-1t}$ diverges in the probability limit when $d > -\frac{1}{2}$.

A2. Appendix 2: Derivation of asymptotic results for regressions with an *I*(0) dependent variable and *I*(1) regressors

To obtain **(28)** start by rewriting equation **(24)** as:

(A2.1)

$$A = \left\{ T\left(T^{-1}\sum_{t=1}^{T}\underline{x}_{1t}\underline{x}_{1t}'\right) - T\left(T^{-1}\sum_{t=1}^{T}\underline{x}_{1t}\underline{x}_{2t}'\right)\left[T^{2}\left(T^{-2}\sum_{t=1}^{T}\underline{x}_{2t}\underline{x}_{2t}'\right)\right]^{-1} T\left(T^{-1}\sum_{t=1}^{T}\underline{x}_{2t}\underline{x}_{1t}'\right)\right\}^{-1}$$

Substitution of **(17)**, **(18)** and **(21)** into **(A2.1)**, after taking the probability limit, gives:

$$\underset{T\to\infty}{p\lim} A = \left\{ T\underline{\Sigma}_{1,0} - T\int \underline{b}_2{}' d\underline{B}_1 \left(T^{2}\int \underline{b}_2\underline{b}_2{}'\right)^{-1} T\int \underline{b}_2 d\underline{B}_1{}' \right\}^{-1} \quad \textbf{(A2.2)}$$

$$\Rightarrow \underset{T\to\infty}{p\lim} A = \frac{1}{T\underline{\Sigma}_{1,0} - T^{2}T^{-2}\int \underline{b}_2{}' d\underline{B}_1\left(\int \underline{b}_2\underline{b}_2{}'\right)^{-1}\int \underline{b}_2 d\underline{B}_1{}'} \quad \textbf{(A2.3)}$$

$$\Rightarrow \underset{T\to\infty}{p\lim} TA = \frac{1}{\underline{\Sigma}_{1,0} - T^{-1}\int \underline{b}_2{}' d\underline{B}_1\left(\int \underline{b}_2\underline{b}_2{}'\right)^{-1}\int \underline{b}_2 d\underline{B}_1{}'} \quad \textbf{(A2.4)}$$

$$\therefore \underset{T\to\infty}{p\lim} TA = \underline{\Sigma}_{1,0}^{-1} \quad \textbf{(2.8)}$$

This is because as $T \to \infty$ so $T^{-1}\int \underline{b}_2{}' d\underline{B}_1\left(\int \underline{b}_2\underline{b}_2{}'\right)^{-1}\int \underline{b}_2 d\underline{B}_1{}' \to \underline{0}$.

To obtain **(29)** start by rewriting equation **(25)** as:

$$B = -T^{-1}(TA)\left\{ T\left(T^{-1}\sum_{t=1}^{T}\underline{x}_{1t}\underline{x}_{2t}'\right)\left[T^{2}\left(T^{-2}\sum_{t=1}^{T}\underline{x}_{2t}\underline{x}_{2t}'\right)\right]^{-1}\right\} \quad \textbf{(A2.5)}$$

Substitution of **(17)**, **(18)** and **(28)** into **(25)**, yields:

$$B \Rightarrow -TT^{-1}T^{-2}\left(\underline{\Sigma}_{1,0}^{-1}\right)\left(\int \underline{b}_2{}' d\underline{B}_1\right)\left(\int \underline{b}_2\underline{b}_2{}'\right)^{-1} \quad \textbf{(A2.6)}$$

$$\therefore T^{2}B \Rightarrow -\underline{\Sigma}_{1,0}^{-1}\int \underline{b}_2{}' d\underline{B}_1\left(\int \underline{b}_2\underline{b}_2{}'\right)^{-1} \quad \textbf{(2.9)}$$

To obtain **(30)** start by rewriting equation **(27)** as:

(A2.7)

$$D = \left[T^2 \left(T^{-2} \sum_{t=1}^{T} \underline{x}_{2t} \underline{x}_{2t}' \right) \right]^{-1} \left\{ \underline{I}_{m_2} + \left[T \left(T^{-1} \sum_{t=1}^{T} \underline{x}_{2t} \underline{x}_{1t}' \right) \right] \left[T^{-1} (TA) \right] \left[T \left(T^{-1} \sum_{t=1}^{T} \underline{x}_{1t} \underline{x}_{2t}' \right) \right] \left[T^2 \left(T^{-2} \sum_{t=1}^{T} \underline{x}_{2t} \underline{x}_{2t}' \right) \right]^{-1} \right\}$$

(A2.8)

$$D \Rightarrow T^{-2} \left(\int \underline{b}_2 \underline{b}_2' \right)^{-1} \left[\underline{I}_{m_2} + TTT^{-1} T^{-2} \left(\int \underline{b}_2 d\underline{B}_1' \right) \left(\underline{\Sigma}_{1,0}^{-1} \right) \left(\int \underline{b}_2' d\underline{B}_1 \right) \left(\int \underline{b}_2 \underline{b}_2' \right)^{-1} \right]$$

(A2.9)

$$D \Rightarrow T^{-2} \left(\int \underline{b}_2 \underline{b}_2' \right)^{-1} + T^{-3} \left(\int \underline{b}_2 \underline{b}_2' \right)^{-1} \left(\int \underline{b}_2 d\underline{B}_1' \right) \left(\underline{\Sigma}_{1,0}^{-1} \right) \left(\int \underline{b}_2' d\underline{B}_1 \right) \left(\int \underline{b}_2 \underline{b}_2' \right)^{-1}$$

(A2.10)

$$T^2 D \Rightarrow \left(\int \underline{b}_2 \underline{b}_2' \right)^{-1} + T^{-1} \left(\int \underline{b}_2 \underline{b}_2' \right)^{-1} \left(\int \underline{b}_2 d\underline{B}_1' \right) \left(\underline{\Sigma}_{1,0}^{-1} \right) \left(\int \underline{b}_2' d\underline{B}_1 \right) \left(\int \underline{b}_2 \underline{b}_2' \right)^{-1}$$

$$\therefore T^2 D \Rightarrow \left(\int \underline{b}_2 \underline{b}_2' \right)^{-1} \tag{30}$$

This is because $\left[T^{-1} \left(\int \underline{b}_2 \underline{b}_2' \right)^{-1} \left(\int \underline{b}_2 d\underline{B}_1' \right) \left(\underline{\Sigma}_{1,0}^{-1} \right) \left(\int \underline{b}_2' d\underline{B}_1 \right) \left(\int \underline{b}_2 \underline{b}_2' \right)^{-1} \right] \to \underline{0}$ as $T \to \infty$.

To obtain **(32)** start by rewriting equation **(31)** as:

$$\underline{\hat{b}}_1 = \left[T^{-1} (TA) \right] \left[T^{0.5} \left(T^{-0.5} \sum_{t=1}^{T} \underline{x}_{1t} y_{0t} \right) \right] + \left[T^{-2} \left(T^2 B \right) \right] \left[T \left(T^{-1} \sum_{t=1}^{T} \underline{x}_{2t} y_{0t} \right) \right] \tag{A2.11}$$

Substitution of **(19)**, **(22)**, **(28)** and **(29)** in **(A2.11)**, after taking the probability limit, yields:

(A2.12)

$$\Rightarrow \underset{T \to \infty}{p \lim} \underline{\hat{b}}_1 = \left[T^{-1} \left(\underline{\Sigma}_{1,0}^{-1} \right) T^{0.5} \underline{\Gamma} \right] + \left\{ T^{-2} T \left[- \underline{\Sigma}_{1,0}^{-1} \int \underline{b}_2' d\underline{B}_1 \left(\int \underline{b}_2 \underline{b}_2' \right)^{-1} \right] \left(\int \underline{b}_2' dB_0 \right) \right\}$$

$$\Rightarrow \underset{T \to \infty}{p \lim} \underline{\hat{b}}_1 = T^{-0.5} \underline{\Sigma}_{1,0}^{-1} \underline{\Gamma} - T^{-1} \underline{\Sigma}_{1,0}^{-1} \int \underline{b}_2' d\underline{B}_1 \left(\int \underline{b}_2 \underline{b}_2' \right)^{-1} \int \underline{b}_2' dB_0 \tag{A2.13}$$

$$\therefore \underset{T \to \infty}{p \lim} T^{0.5} \underline{\hat{b}}_1 = \underline{\Sigma}_{1,0}^{-1} \underline{\Gamma} =: \underline{\mathbf{b}}_1 \tag{3.2}$$

To obtain **(34)** start by rewriting equation **(33)** as:

$$\underline{\hat{b}}_2 = \left[T^{-2} \left(T^2 C \right) \right] \left[T^{0.5} \left(T^{-0.5} \sum_{t=1}^{T} \underline{x}_{1t} y_{0t} \right) \right] + \left[T^{-2} \left(T^2 D \right) \right] \left[T \left(T^{-1} \sum_{t=1}^{T} \underline{x}_{2t} y_{0t} \right) \right] \tag{A2.14}$$

(A2.15)

$$\underline{\hat{b}}_2 \Rightarrow \left[T^{-2}T^{0.5}\left(-\underline{\Sigma}_{1,0}^{-1}\int \underline{b}_2' \, d\underline{B}_1 \left(\int \underline{b}_2\underline{b}_2'\right)^{-1}\right)'\underline{\Gamma}\right] + \left[T^{-2}T\left(\int \underline{b}_2\underline{b}_2'\right)^{-1}\int \underline{b}_2' \, dB_0\right]$$

(A2.16)

$$\underline{\hat{b}}_2 \Rightarrow T^{-1}\left(\int \underline{b}_2\underline{b}_2'\right)^{-1}\int\underline{b}_2' \, dB_0 - \left[T^{-1.5}\left(\underline{\Sigma}_{1,0}^{-1}\int\underline{b}_2' \, d\underline{B}_1\left(\int\underline{b}_2\underline{b}_2'\right)^{-1}\right)'\underline{\Gamma}\right]$$

$$\therefore T\underline{\hat{b}}_2 \Rightarrow \left(\int \underline{b}_2\underline{b}_2'\right)^{-1}\int\underline{b}_2' \, dB_0 =: \underline{\mathbf{b}}_2 \tag{34}$$

To obtain **(36)** start by rewriting equation **(35)** as:

(A2.17)

$$s^2 = T^{-1}\left(\sum_{t=1}^{T} y_{0t}^2 + \sum_{t=1}^{T} \underline{\hat{b}}_1' \underline{x}_{1t}\underline{x}_{1t}'\underline{\hat{b}}_1 + \sum_{t=1}^{T} \underline{\hat{b}}_2'\underline{x}_{2t}\underline{x}_{2t}'\underline{\hat{b}}_2 + 2\sum_{t=1}^{T} \underline{\hat{b}}_1'\underline{x}_{1t}\underline{x}_{2t}'\underline{\hat{b}}_2 - 2\sum_{t=1}^{T} \underline{\hat{b}}_1'\underline{x}_{1t}y_{0t} - 2\sum_{t=1}^{T}\underline{\hat{b}}_2'\underline{x}_{2t}y_{0t}\right)$$

(A2.18)

$$\Rightarrow s^2 = T^{-1}\left\{\begin{array}{l} \left[\left[T\left(T^{-1}\sum_{t=1}^{T} y_{0t}^2\right)\right] + \left[\left(T^{0.5}\underline{\hat{b}}_1'\right)\left(T^{-1}\sum_{t=1}^{T}\underline{x}_{1t}\underline{x}_{1t}'\right)\left(T^{0.5}\underline{\hat{b}}_1\right)\right] + \left[\left(T\underline{\hat{b}}_2'\right)\left(T^{-2}\sum_{t=1}^{T} \underline{x}_{2t}\underline{x}_{2t}'\right)\left(T\underline{\hat{b}}_2\right)\right]\right. \\ \left. + \left[2T^{-0.5}\left(T^{0.5}\underline{\hat{b}}_1'\right)\left(T^{-1}\sum_{t=1}^{T}\underline{x}_{1t}\underline{x}_{2t}'\right)\left(T\underline{\hat{b}}_2\right)\right] - \left[2\left(T^{0.5}\underline{\hat{b}}_1'\right)\left(T^{-0.5}\sum_{t=1}^{T} \underline{x}_{1t}y_{0t}\right)\right] - \left[2\left(T\underline{\hat{b}}_2'\right)\left(T^{-1}\sum_{t=1}^{T}\underline{x}_{2t}y_{0t}\right)\right]\right] \end{array}\right\}$$

(A2.19)

Substitution of **(17)**, **(18)**, **(19)**, **(20)**, **(21)**, **(22)**, **(32)** and **(34)** into **(A2.19)**, after taking the probability limit, gives

$$\Rightarrow \underset{T\to\infty}{p\lim}\, s^2 = T^{-1}\left\{\begin{array}{l} \left[T\,\Sigma_{0,0} + \left[\left(\underline{\Sigma}_{1,0}^{-1}\underline{\Gamma}\right)'\underline{\Sigma}_{1,0}\left(\underline{\Sigma}_{1,0}^{-1}\underline{\Gamma}\right)\right] + \left[\left[\left(\int\underline{b}_2\underline{b}_2'\right)^{-1}\int\underline{b}_2' \, dB_0\right]'\left[\int\underline{b}_2\underline{b}_2'\right]\left[\left(\int\underline{b}_2\underline{b}_2'\right)^{-1}\int\underline{b}_2' \, dB_0\right]\right]\right. \\ \left. + \left[2T^{-0.5}\left[\underline{\Sigma}_{1,0}^{-1}\underline{\Gamma}\right]'\left(\int\underline{b}_2' \, d\underline{B}_1\right)\left[\left(\int\underline{b}_2\underline{b}_2'\right)^{-1}\int\underline{b}_2' \, dB_0\right]\right] - \left[2\left[\underline{\Sigma}_{1,0}^{-1}\underline{\Gamma}\right]'\underline{\Gamma}\right] - \left[2\left[\left(\int\underline{b}_2\underline{b}_2'\right)^{-1}\int\underline{b}_2' \, dB_0\right]'\left(\int\underline{b}_2' \, dB_0\right)\right]\right] \end{array}\right\}$$

(A2.20)

$$\begin{aligned} \Rightarrow \underset{T\to\infty}{p\lim}\, s^2 = \Sigma_{0,0} &+ T^{-1}\underline{\Gamma}'\underline{\Sigma}_{1,0}^{-1}\underline{\Gamma} + T^{-1}\left[\int\underline{b}_2 dB_0'\left(\int\underline{b}_2\underline{b}_2'\right)^{-1}\int\underline{b}_2' \, dB_0\right] + 2T^{-1.5}\underline{\Gamma}'\underline{\Sigma}_{1,0}^{-1}\left(\int\underline{b}_2' \, d\underline{B}_1\right)\left(\int\underline{b}_2\underline{b}_2'\right)^{-1}\int\underline{b}_2' \, dB_0 \\ &- 2T^{-1}\underline{\Gamma}'\underline{\Sigma}_{1,0}^{-1}\underline{\Gamma} - 2T^{-1}\left[\int\underline{b}_2 dB_0'\left(\int\underline{b}_2\underline{b}_2'\right)^{-1}\int\underline{b}_2' \, dB_0\right] \end{aligned}$$

$$\therefore p\lim s^2 = \Sigma_{0,0} \tag{36}$$

To obtain **(38)** start by rewriting equation **(37)** as:

$$s_{\left[\underline{\hat{b}}_1\right]_k} = \left[s^2T^{-1}\left(TA_{kk}\right)\right]^{0.5} \tag{A2.21}$$

Substitution of **(28)** and **(36)** into **(A2.21)**, after taking the probability limit, gives:

$$p\lim s_{[\hat{\underline{b}}_1]_k} = \left[\Sigma_{0,0}\, T^{-1}\left(\underline{\Sigma}_{1,0}\right)^{-1}_{kk}\right]^{0.5} \qquad \textbf{(A2.22)}$$

$$\Rightarrow p\lim s_{[\hat{\underline{b}}_1]_k} = T^{-0.5}\left(\Sigma_{0,0}\right)^{0.5}\left(\underline{\Sigma}_{1,0}\right)^{-0.5}_{kk} \qquad \textbf{(A2.23)}$$

$$\therefore p\lim T^{0.5} s_{[\hat{\underline{b}}_1]_k} = \left(\Sigma_{0,0}\right)^{0.5}\left(\underline{\Sigma}_{1,0}\right)^{-0.5}_{kk} \qquad \textbf{(38)}$$

To obtain **(40)** start by rewriting equation **(39)** as:

$$t_{[\hat{\underline{b}}_1]_k} = \frac{T^{-0.5}\left[T\hat{\underline{b}}_1\right]_k}{T^{-0.5}\left(T^{0.5} s_{[\hat{\underline{b}}_1]_k}\right)} \qquad \textbf{(A2.24)}$$

Substitution of **(32)** and **(38)** into **(A2.24)**, after taking the probability limit, gives:

$$p\lim t_{[\hat{\underline{b}}_1]_k} = \frac{\left[\underline{\mathbf{b}}_1\right]_k}{\left(\Sigma_{0,0}\right)^{0.5}\left(\underline{\Sigma}_{1,0}\right)^{-0.5}_{kk}} \qquad \textbf{(40)}$$

To obtain **(42)** start by rewriting equation **(41)** as:

$$s_{[\hat{\underline{b}}_2]_j} = \left[s^2 T^{-2}\left(T^2 D_{jj}\right)\right]^{0.5} \qquad \textbf{(A2.25)}$$

Substitution of **(30)** and **(36)** into **(A2.25)**, gives:

$$s_{[\hat{\underline{b}}_2]_j} \Rightarrow \left\{\Sigma_{0,0}\, T^{-2}\left[\left(\int \underline{b}_2 \underline{b}_2{}'\right)^{-1}\right]_{jj}\right\}^{0.5} \qquad \textbf{(A2.26)}$$

$$s_{[\hat{\underline{b}}_2]_j} \Rightarrow T^{-1}\left(\Sigma_{0,0}\right)^{0.5}\left[\left(\int \underline{b}_2 \underline{b}_2{}'\right)^{-0.5}\right]_{jj} \qquad \textbf{(A2.27)}$$

$$\therefore T s_{[\hat{\underline{b}}_2]_j} \Rightarrow \left(\Sigma_{0,0}\right)^{0.5}\left[\left(\int \underline{b}_2 \underline{b}_2{}'\right)^{-0.5}\right]_{jj} \qquad \textbf{(42)}$$

To obtain **(44)** start by rewriting equation **(43)** as:

$$t_{[\hat{\underline{b}}_2]_j} = \frac{T^{-1}\left[T\hat{\underline{b}}_2\right]_j}{T^{-1} T s_{[\hat{\underline{b}}_2]_j}} \qquad \textbf{(A2.28)}$$

Substitution of **(34)** and **(42)** into **(A2.28)**, gives:

$$t_{[\hat{\underline{b}}_2]_j} \Rightarrow \frac{[\underline{\mathbf{b}}_2]_j}{(\Sigma_{0,0})^{0.5}\left[\left(\int \underline{b}_2\underline{b}_2'\right)^{-0.5}\right]_{jj}} \tag{44}$$

To obtain **(46)** start by rewriting equation **(45)** as:[55]

$$R^2 = \frac{T^{-0.5}\left(T^{0.5}\hat{\underline{b}}_1'\right)T\left(T^{-1}\sum_{t=1}^{T}\underline{x}_{1t}\underline{x}_{1t}'\right)T^{-0.5}\left(T^{0.5}\hat{\underline{b}}_1\right)+2T^{-1}\left(T\hat{\underline{b}}_2'\right)T\left(T^{-1}\sum_{t=1}^{T}\underline{x}_{2t}\underline{x}_{1t}'\right)T^{-0.5}\left(T^{0.5}\hat{\underline{b}}_1\right)+T^{-1}\left(T\hat{\underline{b}}_2'\right)T^2\left(T^{-2}\sum_{t=1}^{T}\underline{x}_{2t}\underline{x}_{2t}'\right)T^{-1}\left(T\hat{\underline{b}}_2\right)}{T\left(T^{-1}\sum_{t=1}^{T}y_{0t}^2\right)} \tag{A2.29}$$

Substitution of **(17)**, **(18)**, **(20)**, **(21)**, **(32)** and **(34)** into **(A2.29)**, gives:

$$R^2 \Rightarrow \frac{T^{-1}\underline{\mathbf{b}}_1'\underline{\Sigma}_{1,0}\underline{\mathbf{b}}_1 + 2T^{-1.5}\underline{\mathbf{b}}_2'\int \underline{b}_2 d\underline{B}_1'\underline{\mathbf{b}}_1 + T^{-1}\underline{\mathbf{b}}_2'\int \underline{b}_2\underline{b}_2'\underline{\mathbf{b}}_2}{\Sigma_{0,0}} \tag{A2.30}$$

$$R^2 \Rightarrow \frac{T^{-1}\left(\underline{\mathbf{b}}_1'\underline{\Sigma}_{1,0}\underline{\mathbf{b}}_1 + \underline{\mathbf{b}}_2'\int \underline{b}_2\underline{b}_2'\underline{\mathbf{b}}\right)_2}{\Sigma_{0,0}} \tag{A2.31}$$

$$\therefore TR^2 \Rightarrow \frac{\underline{\mathbf{b}}_1'\underline{\Sigma}_{1,0}\underline{\mathbf{b}}_1 + \underline{\mathbf{b}}_2'\int \underline{b}_2\underline{b}_2'\underline{\mathbf{b}}_2}{\Sigma_{0,0}} \tag{46}$$

To obtain **(48)** start by rewriting equation **(47)** as:

$$F_1 = \frac{T^{-0.5}\left(T^{0.5}\hat{\underline{b}}_1'\right)T(TA)^{-1}T^{-0.5}\left(T^{0.5}\hat{\underline{b}}_1\right)}{s^2 m_1} \tag{A2.32}$$

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[55] Equation **(4.5)** derives from: $R^2 = \frac{\sum_{t=1}^{T}\hat{y}_t^2}{\sum_{t=1}^{T}y_{0t}^2} = \frac{\sum_{t=1}^{T}(\hat{\underline{b}}_1'\underline{x}_{1t}+\hat{\underline{b}}_2'\underline{x}_{2t})^2}{\sum_{t=1}^{T}y_{0t}^2} = \frac{\sum_{t=1}^{T}(\hat{\underline{b}}_1'\underline{x}_{1t}+\hat{\underline{b}}_2'\underline{x}_{2t})(\hat{\underline{b}}_1'\underline{x}_{1t}+\hat{\underline{b}}_2'\underline{x}_{2t})'}{\sum_{t=1}^{T}y_{0t}^2}$, $\Rightarrow R^2 = \frac{\hat{\underline{b}}_1'\sum_{t=1}^{T}\underline{x}_{1t}\underline{x}_{1t}'\hat{\underline{b}}_1 + \hat{\underline{b}}_1'\sum_{t=1}^{T}\underline{x}_{1t}\underline{x}_{2t}'\hat{\underline{b}}_2 + \hat{\underline{b}}_2'\sum_{t=1}^{T}\underline{x}_{2t}\underline{x}_{1t}'\hat{\underline{b}}_1 + \hat{\underline{b}}_2'\sum_{t=1}^{T}\underline{x}_{2t}\underline{x}_{2t}'\hat{\underline{b}}_2}{\sum_{t=1}^{T}y_{0t}^2}$, which, given all elements in the equation are scalars, $\Rightarrow R^2 = \frac{\hat{\underline{b}}_1'\sum_{t=1}^{T}\underline{x}_{1t}\underline{x}_{1t}'\hat{\underline{b}}_1 + 2\hat{\underline{b}}_2'\sum_{t=1}^{T}\underline{x}_{2t}\underline{x}_{1t}'\hat{\underline{b}}_1 + \hat{\underline{b}}_2'\sum_{t=1}^{T}\underline{x}_{2t}\underline{x}_{2t}'\hat{\underline{b}}_2}{\sum_{t=1}^{T}y_{0t}^2}$.

Substitution of **(28)**, **(32)** and **(36)** into **(A2.32)**, after taking the probability limit, gives:

$$p\lim F_1 = \frac{\underline{\mathbf{b}}_1' \underline{\Sigma}_{1,0} \underline{\mathbf{b}}_1}{\Sigma_{0,0}\, m_1} \tag{48}$$

To obtain **(50)** start by rewriting equation **(49)** as:

$$F_2 = \frac{T^{-1}\left(T\hat{\underline{b}}_2\right)' T^2 (T^{-2}D)^{-1} T^{-1}\left(T\hat{\underline{b}}_2\right)}{s^2 m_2} \tag{A2.33}$$

Substitution of **(30)**, **(34)** and **(36)** into **(A2.33)**, gives:

$$F_2 \Rightarrow \frac{\underline{\mathbf{b}}_2' \int \underline{b}_2 \underline{b}_2' \underline{\mathbf{b}}_2}{\Sigma_{0,0}\, m_2} \tag{50}$$

To obtain **(52)** start by rewriting equation **(51)** as:[56]

(A2.34)

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[56] The Durbin-Watson statistic is defined as: $DW = \frac{\sum_{t=2}^{T} (\hat{u}_{1t} - \hat{u}_{1t-1})^2}{\sum_{t=1}^{T} \hat{u}_{1t}^2} = \frac{T^{-1}\sum_{t=2}^{T} (\hat{u}_{1t} - \hat{u}_{1t-1})^2}{T^{-1}\sum_{t=1}^{T} \hat{u}_{1t}^2}$.

Substitution of $s^2 = T^{-1}\sum_{t=1}^{T}\hat{u}_{1t}^2$ and $\hat{u}_{1t} = y_{0t} - \hat{\underline{b}}_1' \underline{x}_{1t} - \hat{\underline{b}}_2' \underline{x}_{2t}$ into DW gives,

$$DW = \frac{T^{-1}\sum_{t=2}^{T}\left[\left(y_{0t} - \hat{\underline{b}}_1' \underline{x}_{1t} - \hat{\underline{b}}_2' \underline{x}_{2t}\right) - \left(y_{0t-1} - \hat{\underline{b}}_1' \underline{x}_{1t-1} - \hat{\underline{b}}_2' \underline{x}_{2t-1}\right)\right]^2}{s^2}.$$

$$DW = \frac{T^{-1}\sum_{t=2}^{T}\left[(y_{0t} - y_{0t-1}) - \hat{\underline{b}}_1'(\underline{x}_{1t} - \underline{x}_{1t-1}) - \hat{\underline{b}}_2'(\underline{x}_{2t} - \underline{x}_{2t-1})\right]^2}{s^2}.$$

From **(4)**, $\underline{x}_{2t} - \underline{x}_{2t-1} = \left(\underline{X}_{2t} - \overline{\underline{X}}_2\right) - \left(\underline{X}_{2t-1} - \overline{\underline{X}}_2\right) = \underline{X}_{2t} - \underline{X}_{2t-1} = \underline{v}_{2t}$ – although $\underline{X}_{2t} \sim I(1)$, because it is a random walk without drift its mean is constant through time and, asymptotically, the contemporaneous and lagged mean are approximately equal. Also, from **(2)** and **(3)**, the following hold (approximately) in large samples $y_{0t} - y_{0t-1} = \left(Y_{0t} - \overline{Y}_0\right) - \left(Y_{0t-1} - \overline{Y}_0\right) = v_{0t} - v_{0t-1}$ and $\underline{x}_{1t} - \underline{x}_{1t-1} = \left(\underline{X}_{1t} - \overline{\underline{X}}_1\right) - \left(\underline{X}_{1t-1} - \overline{\underline{X}}_1\right) = \underline{X}_{1t} - \underline{X}_{1t-1} = \underline{v}_{1t} - \underline{v}_{1t-1}$. Substitution of these terms into DW gives **(51)**, thus: $DW = \frac{T^{-1}\sum_{t=2}^{T}\left[(v_{0t} - v_{0t-1}) - \hat{\underline{b}}_1'(\underline{v}_{1t} - \underline{v}_{1t-1}) - \hat{\underline{b}}_2' \underline{v}_{2t}\right]^2}{s^2}$. This expression is different from that given in Hassler (1996) due to the dependent variable being stationary in this paper and nonstationary in his.

$$DW=\frac{T^{-1}\sum_{t=2}^{T}\left[\left(v_{0t}-v_{0t-1}\right)-\underline{\hat{b}}_1'\left(\underline{v}_{1t}-\underline{v}_{1t-1}\right)-\underline{\hat{b}}_2'\underline{v}_{2t}\right]\left[\left(v_{0t}-v_{0t-1}\right)-\underline{\hat{b}}_1'\left(\underline{v}_{1t}-\underline{v}_{1t-1}\right)-\underline{\hat{b}}_2'\underline{v}_{2t}\right]'}{s^2}$$

(A2.35)

$$\Rightarrow DW=\frac{T^{-1}\left[\begin{array}{l}\sum_{t=2}^{T}\left(v_{0t}-v_{0t-1}\right)\left(v_{0t}-v_{0t-1}\right)'+\underline{\hat{b}}_1'\sum_{t=2}^{T}\left(\underline{v}_{1t}-\underline{v}_{1t-1}\right)\left(\underline{v}_{1t}-\underline{v}_{1t-1}\right)'\underline{\hat{b}}_1+\underline{\hat{b}}_2'\sum_{t=2}^{T}\underline{v}_{2t}\underline{v}_{2t}'\underline{\hat{b}}_2\\-\sum_{t=2}^{T}\left(v_{0t}-v_{0t-1}\right)\left(\underline{v}_{1t}-\underline{v}_{1t-1}\right)'\underline{\hat{b}}_1-\underline{\hat{b}}_1'\sum_{t=2}^{T}\left(\underline{v}_{1t}-\underline{v}_{1t-1}\right)\left(v_{0t}-v_{0t-1}\right)'-\sum_{t=2}^{T}\left(v_{0t}-v_{0t-1}\right)\underline{v}_{2t}'\underline{\hat{b}}_2\\-\underline{\hat{b}}_2'\sum_{t=2}^{T}\underline{v}_{2t}\left(v_{0t}-v_{0t-1}\right)'+\underline{\hat{b}}_1'\sum_{t=2}^{T}\left(\underline{v}_{1t}-\underline{v}_{1t-1}\right)\underline{v}_{2t}'\underline{\hat{b}}_2+\underline{\hat{b}}_2'\sum_{t=2}^{T}\underline{v}_{2t}\left(\underline{v}_{1t}-\underline{v}_{1t-1}\right)'\underline{\hat{b}}_1\end{array}\right]}{s^2}$$

Because all of the elements are scalars this can be re-written as:

(A2.36)

$$\Rightarrow DW=\frac{T^{-1}\left[\begin{array}{l}\sum_{t=2}^{T}\left(v_{0t}-v_{0t-1}\right)\left(v_{0t}-v_{0t-1}\right)'+\underline{\hat{b}}_1'\sum_{t=2}^{T}\left(\underline{v}_{1t}-\underline{v}_{1t-1}\right)\left(\underline{v}_{1t}-\underline{v}_{1t-1}\right)'\underline{\hat{b}}_1+\underline{\hat{b}}_2'\sum_{t=2}^{T}\underline{v}_{2t}\underline{v}_{2t}'\underline{\hat{b}}_2\\-2\underline{\hat{b}}_1'\sum_{t=2}^{T}\left(\underline{v}_{1t}-\underline{v}_{1t-1}\right)\left(v_{0t}-v_{0t-1}\right)'-2\underline{\hat{b}}_2'\sum_{t=2}^{T}\underline{v}_{2t}\left(v_{0t}-v_{0t-1}\right)'+2\underline{\hat{b}}_1'\sum_{t=2}^{T}\left(\underline{v}_{1t}-\underline{v}_{1t-1}\right)\underline{v}_{2t}'\underline{\hat{b}}_2\end{array}\right]}{s^2}$$

(A2.37)

$$\Rightarrow DW=\frac{T^{-1}\left[\begin{array}{l}\sum_{t=2}^{T}v_{0t}v_{0t}-2\sum_{t=2}^{T}v_{0t}v_{0t-1}+\sum_{t=2}^{T}v_{0t-1}v_{0t-1}+\underline{\hat{b}}_1'\sum_{t=2}^{T}\underline{v}_{1t}\underline{v}_{1t}'\underline{\hat{b}}_1-2\underline{\hat{b}}_1'\sum_{t=2}^{T}\underline{v}_{1t}\underline{v}_{1t-1}'\underline{\hat{b}}_1+\underline{\hat{b}}_1'\sum_{t=2}^{T}\underline{v}_{1t-1}\underline{v}_{1t-1}'\underline{\hat{b}}_1\\+\underline{\hat{b}}_2'\sum_{t=2}^{T}\underline{v}_{2t}\underline{v}_{2t}'\underline{\hat{b}}_2-2\underline{\hat{b}}_1'\sum_{t=2}^{T}\underline{v}_{1t}v_{0t}+2\underline{\hat{b}}_1'\sum_{t=2}^{T}\underline{v}_{1t}v_{0t-1}+2\underline{\hat{b}}_1'\sum_{t=2}^{T}\underline{v}_{1t-1}v_{0t}-2\underline{\hat{b}}_1'\sum_{t=2}^{T}\underline{v}_{1t-1}v_{0t-1}\\-2\underline{\hat{b}}_2'\sum_{t=2}^{T}\underline{v}_{2t}v_{0t}+2\underline{\hat{b}}_2'\sum_{t=2}^{T}\underline{v}_{2t}v_{0t-1}+2\underline{\hat{b}}_1'\sum_{t=2}^{T}\underline{v}_{1t}\underline{v}_{2t}'\underline{\hat{b}}_2-2\underline{\hat{b}}_1'\sum_{t=2}^{T}\underline{v}_{1t-1}\underline{v}_{2t}'\underline{\hat{b}}_2\end{array}\right]}{s^2}$$

Re-arranging and recognising that asymptotically $\sum_{t=2}^{T}v_{0t}v_{0t}=\sum_{t=2}^{T}v_{0t-1}v_{0t-1}$ and $\sum_{t=2}^{T}\underline{v}_{1t}\underline{v}_{1t}'=\sum_{t=2}^{T}\underline{v}_{1t-1}\underline{v}_{1t-1}'$ yields:

(A2.37)

$$\Rightarrow DW=\frac{\left[\begin{array}{l}2\left(T^{-1}\sum_{t=2}^{T}v_{0t}v_{0t}-T^{-1}\sum_{t=2}^{T}v_{0t}v_{0t-1}\right)+2T^{-1}\left(T^{0.5}\underline{\hat{b}}_1'\right)\left(T^{-1}\sum_{t=2}^{T}\underline{v}_{1t}\underline{v}_{1t}'-T^{-1}\sum_{t=2}^{T}\underline{v}_{1t}\underline{v}_{1t-1}'\right)\left(T^{0.5}\underline{\hat{b}}_1\right)+T^{-2}\left(T\underline{\hat{b}}_2'\right)\left(T^{-1}\sum_{t=2}^{T}\underline{v}_{2t}\underline{v}_{2t}'\right)\left(T\underline{\hat{b}}_2\right)\\-2T^{-1}\left(T^{0.5}\underline{\hat{b}}_1'\right)\left(T^{-0.5}\sum_{t=2}^{T}\underline{v}_{1t}v_{0t}\right)+2T^{-1}\left(T^{0.5}\underline{\hat{b}}_1'\right)\left(T^{-0.5}\sum_{t=2}^{T}\underline{v}_{1t}v_{0t-1}\right)+2T^{-1}\left(T^{0.5}\underline{\hat{b}}_1'\right)\left(T^{-0.5}\sum_{t=2}^{T}\underline{v}_{1t-1}v_{0t}\right)-2T^{-1}\left(T^{0.5}\underline{\hat{b}}_1'\right)\left(T^{-0.5}\sum_{t=2}^{T}\underline{v}_{1t-1}v_{0t-1}\right)\\-2T^{-1.5}\left(T\underline{\hat{b}}_2'\right)\left(T^{-0.5}\sum_{t=2}^{T}\underline{v}_{2t}v_{0t}\right)+2T^{-1.5}\left(T\underline{\hat{b}}_2'\right)\left(T^{-0.5}\sum_{t=2}^{T}\underline{v}_{2t}v_{0t-1}\right)+2T^{-2}\left(T^{0.5}\underline{\hat{b}}_1'\right)\left(T^{-0.5}\sum_{t=2}^{T}\underline{v}_{1t}\underline{v}_{2t}'\right)\left(T\underline{\hat{b}}_2\right)-2T^{-2}\left(T^{0.5}\underline{\hat{b}}_1'\right)\left(T^{-0.5}\sum_{t=2}^{T}\underline{v}_{1t-1}\underline{v}_{2t}'\right)\left(T\underline{\hat{b}}_2\right)\end{array}\right]}{s^2}$$

Given that v_{0t} and $\underline{v}_{1t}$ are zero mean and stationary homoscedastic processes (where it is assumed that v_{2t} follows a stationary first-order autoregressive process) then, by analogy to **(22)**, we assume that $p\lim T^{-0.5}\sum_{t=2}^{T}\underline{v}_{1t}v_{0t-1}=\underline{\Gamma}_{1,01}$, $p\lim T^{-0.5}\sum_{t=2}^{T}\underline{v}_{1t-1}v_{0t}=\underline{\Gamma}_{0,11}$, $p\lim T^{-0.5}\sum_{t=2}^{T}\underline{v}_{1t-1}v_{0t-1}=\underline{\Gamma}_{01,11}$,

$p\lim T^{-0.5}\sum_{t=2}^{T}\underline{v}_{2t}v_{0t}{'} = \underline{\Gamma}_{0,2}$, $p\lim T^{-0.5}\sum_{t=2}^{T}\underline{v}_{2t}v_{0t-1}{'} = \underline{\Gamma}_{2,01}$, $p\lim T^{-0.5}\sum_{t=2}^{T}\underline{v}_{1t}\underline{v}_{2t}{'} = \underline{\Gamma}_{1,2}$, $p\lim T^{-0.5}\sum_{t=2}^{T}\underline{v}_{1t-1}\underline{v}_{2t}{'} = \underline{\Gamma}_{2,11}$. Substitution of these terms, as well as **(11)**, **(12)**, **(22)**, **(32)** and **(34)** into **(A2.37)**, gives:

(A2.38)

$$\Rightarrow p\lim DW = \frac{\begin{bmatrix} 2\left(\Sigma_{0,0}-\Sigma_{0,1}\right)+2T^{-2}\underline{\mathbf{b}}_1{'}\left(\underline{\Sigma}_{1,0}-\underline{\Sigma}_{1,1}\right)\underline{\mathbf{b}}_1+T^{-2}\underline{\mathbf{b}}_2{'}\Sigma_{2,0}\underline{\mathbf{b}}_2{'}-2T^{-1}\underline{\mathbf{b}}_1{'}\underline{\Gamma}+2T^{-1}\underline{\mathbf{b}}_1{'}\underline{\Gamma}_{1,01}+2T^{-1}\underline{\mathbf{b}}_1{'}\underline{\Gamma}_{0,11} \\ -2T^{-1}\underline{\mathbf{b}}_1{'}\underline{\Gamma}_{01,11}-2T^{-1.5}\underline{\mathbf{b}}_2{'}\underline{\Gamma}_{0,2}+2T^{-1.5}\underline{\mathbf{b}}_2{'}\underline{\Gamma}_{2,01}+2T^{-2}\underline{\mathbf{b}}_1{'}\underline{\Gamma}_{1,2}\underline{\mathbf{b}}_2-2T^{-2}\underline{\mathbf{b}}_1{'}\underline{\Gamma}_{2,11}\underline{\mathbf{b}}_2 \end{bmatrix}}{\Sigma_{0,0}}$$

$$\Rightarrow p\lim DW = \frac{2\left(\Sigma_{0,0}-\Sigma_{0,1}\right)}{\Sigma_{0,0}} \quad \textbf{(A2.39)}$$

$$\therefore p\lim DW = 2\left[1-\left(\frac{\Sigma_{0,1}}{\Sigma_{0,0}}\right)\right] \quad \textbf{(52)}$$

To obtain **(60)** start by rewriting equation **(59)** as:

$$\hat{a} = T^{-0.5}\left(T^{0.5}\overline{Y}_0\right)-T^{-0.5}\left(T^{0.5}\underline{\hat{b}}_1{'}\right)T^{-0.5}\left(T^{0.5}\underline{\overline{X}}_1\right)-T^{-1}\left(T\underline{\hat{b}}_2{'}\right)T^{0.5}\left(T^{-0.5}\underline{\overline{X}}_2\right) \quad \textbf{(A2.40)}$$

Substitution of **(32)**, **(34)**, **(56)**, **(57)** and **(58)** into **(A2.40)**, gives:

$$\hat{a} \Rightarrow T^{-0.5}B_0(1)-T^{-1}\underline{\mathbf{b}}_1{'}\underline{B}_1(1)-T^{-0.5}\underline{\mathbf{b}}_2{'}\int\underline{B}_2 \quad \textbf{(A2.41)}$$

$$\therefore T^{0.5}\hat{a} \Rightarrow B_0(1)-\underline{\mathbf{b}}_2{'}\int\underline{B}_2 \quad \textbf{(60)}$$

To obtain **(63)** start by rewriting equation **(62)** as:

$$\underline{\tilde{b}}_2 = \left[T^{-2}\left(T^{-2}\sum_{t=1}^{T}\underline{x}_{2t}\underline{x}_{2t}{'}\right)^{-1}\right]\left[T\left(T^{-1}\sum_{t=1}^{T}\underline{x}_{2t}y_{0t}\right)\right] \quad \textbf{(A2.42)}$$

Substitution of **(18)** and **(19)** into**(A2.42)**, yields:

$$\underline{\tilde{b}}_2 \Rightarrow T^{-1}\left(\int\underline{b}_2\underline{b}_2{'}\right)^{-1}\int\underline{b}_2{'}dB_0 \quad \textbf{(A2.43)}$$

$$\therefore T\underline{\tilde{b}}_2 \Rightarrow \left(\int\underline{b}_2\underline{b}_2{'}\right)^{-1}\int\underline{b}_2{'}dB_0 = \underline{\mathbf{b}}_2^* \quad \textbf{(63)}$$

To obtain **(65)** start by rewriting equation **(64)** as:

$$\tilde{s}^2 = T^{-1}\sum_{t=1}^{T}\left(y_{0t} - \tilde{\underline{b}}_2{}'\underline{x}_{2t}\right)\left(y_{0t} - \tilde{\underline{b}}_2{}'\underline{x}_{2t}\right)' \tag{A2.44}$$

$$\Rightarrow \tilde{s}^2 = T^{-1}\left(\sum_{t=1}^{T} y_{0t}^2 - 2\tilde{\underline{b}}_2{}'\sum_{t=1}^{T}\underline{x}_{2t}y_{0t} + \tilde{\underline{b}}_2{}'\sum_{t=1}^{T}\underline{x}_{2t}\underline{x}_{2t}{}'\tilde{\underline{b}}_2\right) \tag{A2.45}$$

$$\tilde{s}^2 = T^{-1}\sum_{t=1}^{T} y_{0t}^2 - 2T^{-1}\left[T^{-1}\left(T\tilde{\underline{b}}_2{}'\right)\right]\left[T\left(T^{-1}\sum_{t=1}^{T}\underline{x}_{2t}y_{0t}\right)\right] + T^{-1}\left[T^{-1}\left(T\tilde{\underline{b}}_2{}'\right)\right]\left[T^2\left(T^{-2}\sum_{t=1}^{T}\underline{x}_{2t}\underline{x}_{2t}{}'\right)\right]\left[T^{-1}\left(T\tilde{\underline{b}}_2\right)\right] \tag{A2.46}$$

Substitution of **(18)**, **(19)**, **(20)** and **(63)** into **(A2.46)**, after taking the probability limit, gives:

$$p\lim \tilde{s}^2 = \Sigma_{0,0} - 2T^{-1}\underline{\mathbf{b}}_2^{*\prime}\int \underline{b}_2{}' dB_0 + T^{-1}\underline{\mathbf{b}}_2^{*\prime}\int \underline{b}_2\underline{b}_2{}'\underline{\mathbf{b}}_2^{*} \tag{A2.47}$$

$$\therefore\ p\lim \tilde{s}^2 = \Sigma_{0,0} \tag{65}$$

To obtain **(67)** start by rewriting equation **(66)** as:

$$s_{[\tilde{\underline{b}}_2]_j} = \left\{\tilde{s}^2\left[T^2\left(T^{-2}\sum_{t=1}^{T}\underline{x}_{2t}\underline{x}_{2t}{}'\right)_{jj}\right]^{-1}\right\}^{0.5} \tag{A2.48}$$

Substitution of **(18)** and **(65)** into**(A2.48)**, yields:

$$s_{[\tilde{\underline{b}}_2]_j} \Rightarrow \left\{\Sigma_{0,0}\left[T^2\left(\int \underline{b}_2\underline{b}_2{}'\right)_{jj}\right]^{-1}\right\}^{0.5} \tag{A2.49}$$

$$s_{[\tilde{\underline{b}}_2]_j} \Rightarrow T^{-1}\Sigma_{0,0}^{0.5}\left(\int \underline{b}_2\underline{b}_2{}'\right)_{jj}^{-0.5} \tag{A2.50}$$

$$\therefore Ts_{[\tilde{\underline{b}}_2]_j} \Rightarrow \Sigma_{0,0}^{0.5}\left(\int \underline{b}_2\underline{b}_2{}'\right)_{jj}^{-0.5} \tag{67}$$

To obtain **(69)** start by rewriting equation **(68)** as:

$$\tilde{t}_j = \frac{T^{-1}\left(T\left[\tilde{\underline{b}}_2\right]_j\right)}{T^{-1}\left(Ts_{[\tilde{\underline{b}}_2]_j}\right)} \tag{A2.51}$$

Substitution of **(63)** and **(67)** into this expression, gives:

$$\tilde{t}_j \Rightarrow \frac{T^{-1}\left[\underline{\mathbf{b}}_2^*\right]_j}{T^{-1}\Sigma_{0,0}^{0.5}\left(\int \underline{b}_2\underline{b}_2{}'\right)_{jj}^{-0.5}} \tag{A2.52}$$

$$\therefore \tilde{t}_j \Rightarrow \frac{\left[\underline{\mathbf{b}}_2^*\right]_j}{\Sigma_{0,0}^{0.5}\left(\int \underline{b}_2\underline{b}_2{}'\right)_{jj}^{-0.5}} \tag{69}$$

To obtain **(71)** start by rewriting equation **(70)** as:[57]

$$R^2 = \frac{T^{-1}\left(T\underline{\tilde{b}}_2{}'\right)T^2\left(T^{-2}\sum_{t=1}^{T}\underline{x}_{2t}\underline{x}_{2t}{}'\right)T^{-1}\left(T\underline{\tilde{b}}_2\right)}{T\left(T^{-1}\sum_{t=1}^{T}y_{0t}^2\right)} \tag{A2.53}$$

Substitution of **(18)**, **(20)** and **(63)** into**(A2.53)**, gives:

$$R^2 \Rightarrow \frac{T^{-1}\underline{\mathbf{b}}_2^{*\prime}\int \underline{b}_2\underline{b}_2{}'\,\underline{\mathbf{b}}_2^*}{\Sigma_{0,0}} \tag{A2.54}$$

$$\therefore TR^2 \Rightarrow \frac{\underline{\mathbf{b}}_2^{*\prime}\int \underline{b}_2\underline{b}_2{}'\,\underline{\mathbf{b}}_2^*}{\Sigma_{0,0}} \tag{71}$$

To obtain **(73)** start by rewriting equation **(72)** as:

$$\tilde{F}_2 = \left(\frac{T-m_2-1}{m_2}\right)\left[\frac{T^{-1}\left(TR^2\right)}{1-T^{-1}\left(TR^2\right)}\right] \tag{A2.55}$$

Substitution of **(71)** into **(A2.55)**, gives:

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[57] Equation **(70)** derives from: $R^2 = \frac{\sum_{t=1}^{T}\hat{y}_t^2}{\sum_{t=1}^{T}y_{0t}^2} = \frac{\sum_{t=1}^{T}(\underline{\tilde{b}}_2{}'\underline{x}_{2t})^2}{\sum_{t=1}^{T}y_{0t}^2} = \frac{\sum_{t=1}^{T}(\underline{\tilde{b}}_2{}'\underline{x}_{2t})(\underline{\tilde{b}}_2{}'\underline{x}_{2t})'}{\sum_{t=1}^{T}y_{0t}^2}$,

$\Rightarrow R^2 = \frac{\underline{\tilde{b}}_2{}'\sum_{t=1}^{T}\underline{x}_{2t}\underline{x}_{2t}{}'\,\underline{\tilde{b}}_2}{\sum_{t=1}^{T}y_{0t}^2}$.

$$\tilde{F}_2 \Rightarrow \left(\frac{T - m_2 - 1}{m_2} \right) \left[\frac{T^{-1} \left(\dfrac{\underline{\mathbf{b}}_2^{*\prime} \int \underline{b}_2 \underline{b}_2^{\prime} \underline{\mathbf{b}}_2^{*}}{\Sigma_{0,0}} \right)}{1 - T^{-1} \left(\dfrac{\underline{\mathbf{b}}_2^{*\prime} \int \underline{b}_2 \underline{b}_2^{\prime} \underline{\mathbf{b}}_2^{*}}{\Sigma_{0,0}} \right)} \right] \tag{\textbf{A2.56}}$$

$$\tilde{F}_2 \Rightarrow \frac{\left[T^{-1} (T - m_2 - 1) \right] \left(\dfrac{\underline{\mathbf{b}}_2^{*\prime} \int \underline{b}_2 \underline{b}_2^{\prime} \underline{\mathbf{b}}_2^{*}}{m_2 \Sigma_{0,0}} \right)}{1 - T^{-1} \left(\dfrac{\underline{\mathbf{b}}_2^{*\prime} \int \underline{b}_2 \underline{b}_2^{\prime} \underline{\mathbf{b}}_2^{*}}{\Sigma_{0,0}} \right)} \tag{\textbf{A2.57}}$$

$$\tilde{F}_2 \Rightarrow \frac{1 \times \left(\dfrac{\underline{\mathbf{b}}_2^{*\prime} \int \underline{b}_2 \underline{b}_2^{\prime} \underline{\mathbf{b}}_2^{*}}{m_2 \Sigma_{0,0}} \right)}{1 - 0} \tag{\textbf{A2.58}}$$

Equation **(A2.58)** is obtained from **(A2.57)** because, as the sample size tends to infinity, $\left[T^{-1} (T - m_2 - 1) \right] \to 1$ and $T^{-1} \left(\dfrac{\underline{\mathbf{b}}_2^{*\prime} \int \underline{b}_2 \underline{b}_2^{\prime} \underline{\mathbf{b}}_2^{*}}{\Sigma_{0,0}} \right) \to 0$.

$$\therefore \tilde{F}_2 \Rightarrow \frac{\underline{\mathbf{b}}_2^{*\prime} \int \underline{b}_2 \underline{b}_2^{\prime} \underline{\mathbf{b}}_2^{*}}{m_2 \Sigma_{0,0}} \tag{\textbf{73}}$$

To obtain **(75)** start by rewriting equation **(74)** as:[58]

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[58] The Durbin-Watson statistic is defined as: $DW = \dfrac{\sum_{t=2}^{T} (\hat{u}_{2t} - \hat{u}_{2t-1})^2}{\sum_{t=1}^{T} \hat{u}_{2t}^2} = \dfrac{T^{-1} \sum_{t=2}^{T} (\hat{u}_{2t} - \hat{u}_{2t-1})^2}{T^{-1} \sum_{t=1}^{T} \hat{u}_{2t}^2}$. Substitution of $\tilde{s}^2 = T^{-1} \sum_{t=1}^{T} \hat{u}_{2t}^2$ and $\hat{u}_{2t} = y_{0t} - \hat{\underline{b}}_2^{\prime} \underline{x}_{2t}$ into DW gives, $DW = \dfrac{T^{-1} \sum_{t=2}^{T} \left[\left(y_{0t} - \hat{\underline{b}}_2^{\prime} \underline{x}_{2t} \right) - \left(y_{0t-1} - \hat{\underline{b}}_2^{\prime} \underline{x}_{2t-1} \right) \right]^2}{\tilde{s}^2} = \dfrac{T^{-1} \sum_{t=2}^{T} \left[(y_{0t} - y_{0t-1}) - \hat{\underline{b}}_2^{\prime} (\underline{x}_{2t} - \underline{x}_{2t-1}) \right]^2}{\tilde{s}^2}$. From **(4)**, $\underline{x}_{2t} - \underline{x}_{2t-1} = \left(\underline{X}_{2t} - \overline{\underline{X}}_2 \right) - \left(\underline{X}_{2t-1} - \overline{\underline{X}}_2 \right) = \underline{X}_{2t} - \underline{X}_{2t-1} = \underline{v}_{2t}$ – although $\underline{X}_{2t} \sim I(1)$, because it is a random walk without drift its mean is constant through time and, asymptotically, the contemporaneous and lagged mean are approximately equal. Also, from **(2)**, the following holds (approximately) in large

$$DW = \frac{T^{-1}\sum_{t=2}^{T}\left[\left(v_{0t}-v_{0t-1}\right)-\underline{\tilde{b}}_2^{\prime}\,\underline{v}_{2t}\right]\left[\left(v_{0t}-v_{0t-1}\right)-\underline{\tilde{b}}_2^{\prime}\,\underline{v}_{2t}\right]^{\prime}}{\tilde{s}^2} \qquad \textbf{(A2.59)}$$

$$\textbf{(A2.60)}$$

$$\Rightarrow DW = \frac{T^{-1}\left[\sum_{t=2}^{T}\left(v_{0t}-v_{0t-1}\right)\left(v_{0t}-v_{0t-1}\right)^{\prime}+\underline{\tilde{b}}_2^{\prime}\sum_{t=2}^{T}\underline{v}_{2t}\underline{v}_{2t}^{\prime}\,\underline{\tilde{b}}_2-\sum_{t=2}^{T}\left(v_{0t}-v_{0t-1}\right)\underline{v}_{2t}^{\prime}\,\underline{\tilde{b}}_2-\underline{\tilde{b}}_2^{\prime}\sum_{t=2}^{T}\underline{v}_{2t}\left(v_{0t}-v_{0t-1}\right)^{\prime}\right]}{\tilde{s}^2}$$

Because all of the elements are scalars this can be re-written as:

$$\textbf{(A2.61)}$$

$$DW = \frac{T^{-1}\left[\sum_{t=2}^{T}\left(v_{0t}-v_{0t-1}\right)\left(v_{0t}-v_{0t-1}\right)^{\prime}+\underline{\tilde{b}}_2^{\prime}\sum_{t=2}^{T}\underline{v}_{2t}\underline{v}_{2t}^{\prime}\,\underline{\tilde{b}}_2-2\underline{\tilde{b}}_2^{\prime}\sum_{t=2}^{T}\underline{v}_{2t}\left(v_{0t}-v_{0t-1}\right)^{\prime}\right]}{\tilde{s}^2}$$

$$\textbf{(A2.62)}$$

$$\Rightarrow DW = \frac{T^{-1}\left[\sum_{t=2}^{T}v_{0t}v_{0t}-2\sum_{t=2}^{T}v_{0t}v_{0t-1}+\sum_{t=2}^{T}v_{0t-1}v_{0t-1}+\underline{\tilde{b}}_2^{\prime}\sum_{t=2}^{T}\underline{v}_{2t}\underline{v}_{2t}^{\prime}\,\underline{\tilde{b}}_2-2\underline{\tilde{b}}_2^{\prime}\sum_{t=2}^{T}\underline{v}_{2t}v_{0t}+2\underline{\tilde{b}}_2^{\prime}\sum_{t=2}^{T}\underline{v}_{2t}v_{0t-1}\right]}{\tilde{s}^2}$$

Re-arranging and recognising that asymptotically $\sum_{t=2}^{T}v_{0t}v_{0t}=\sum_{t=2}^{T}v_{0t-1}v_{0t-1}$ yields:

$$\textbf{(A2.63)}$$

$$DW = \frac{\left[\begin{array}{l} 2\left(T^{-1}\sum_{t=2}^{T}v_{0t}v_{0t}-T^{-1}\sum_{t=2}^{T}v_{0t}v_{0t-1}\right)+T^{-2}\left(T\underline{\tilde{b}}_2^{\prime}\right)\left(T^{-1}\sum_{t=2}^{T}\underline{v}_{2t}\underline{v}_{2t}^{\prime}\right)\left(T\underline{\tilde{b}}_2\right) \\ -2T^{-1.5}\left(T\underline{\tilde{b}}_2^{\prime}\right)\left(T^{-0.5}\sum_{t=2}^{T}\underline{v}_{2t}v_{0t}\right)+2T^{-1.5}\left(T\underline{\tilde{b}}_2^{\prime}\right)\left(T^{-0.5}\sum_{t=2}^{T}\underline{v}_{2t}v_{0t-1}\right) \end{array}\right]}{\tilde{s}^2}$$

Substitution of **(11)**, **(12)**, **(63)** and **(65)** into **(A2.63)** and, given that v_{0t} is a zero mean and stationary homoscedastic processes (where it is assumed that v_{2t} follows a stationary first-order autoregressive process) then, by analogy to **(22)**, we assume that, $p\lim T^{-0.5}\sum_{t=2}^{T}\underline{v}_{2t}v_{0t}^{\prime}=\underline{\Gamma}_{0,2}$, $p\lim T^{-0.5}\sum_{t=2}^{T}\underline{v}_{2t}v_{0t-1}^{\prime}=\underline{\Gamma}_{2,01}$. Substitution of these, along with **(11)**, **(12)**, **(63)** and **(65)** into **(A2.63)**, gives:

$$p\lim DW = \frac{\left[2\left(\Sigma_{0,0}-\Sigma_{0,1}\right)+T^{-2}\underline{\mathbf{b}}_2^{*\prime}\,\Sigma_{2,0}\,\underline{\mathbf{b}}_2^{*}-2T^{-1.5}\underline{\mathbf{b}}_2^{*\prime}\,\underline{\Gamma}_{0,2}+2T^{-1.5}\underline{\mathbf{b}}_2^{*\prime}\,\underline{\Gamma}_{2,01}\right]}{\Sigma_{0,0}} \qquad \textbf{(A2.64)}$$

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samples $y_{0t}-y_{0t-1}=\left(Y_{0t}-\bar{Y}_0\right)-\left(Y_{0t-1}-\bar{Y}_0\right)=v_{0t}-v_{0t-1}$. Substitution of these error terms into DW gives equation **(74)**: $DW=\frac{T^{-1}\sum_{t=2}^{T}\left[\left(v_{0t}-v_{0t-1}\right)-\underline{\tilde{b}}_2^{\prime}\,\underline{v}_{2t}\right]^2}{\tilde{s}^2}$.

$$\Rightarrow p\lim DW = \frac{2\left(\Sigma_{0,0} - \Sigma_{0,1}\right)}{\Sigma_{0,0}} \qquad \textbf{(A2.65)}$$

$$\therefore p\lim DW = 2\left[1 - \left(\frac{\Sigma_{0,1}}{\Sigma_{0,0}}\right)\right] \qquad \textbf{(75)}$$

To obtain **(78)** start by rewriting equation **(77)** as:

$$\hat{a} = T^{-0.5}\left(T^{0.5}\overline{Y}_{0}\right) - T^{-1}\left(T\tilde{\underline{b}}_{2}{}'\right)T^{0.5}\left(T^{-0.5}\overline{\underline{X}}_{2}\right) \qquad \textbf{(A2.66)}$$

Substitution of **(63)**, **(56)** and **(58)** into **(A2.66)** gives:

$$\hat{a} \Rightarrow T^{-0.5}B_{0}(1) - T^{-0.5}\underline{\mathbf{b}}_{2}^{*\prime}\int\underline{B}_{2} \qquad \textbf{(A2.67)}$$

$$\therefore T^{0.5}\hat{a} \Rightarrow B_{0}(1) - \underline{\mathbf{b}}_{2}^{*\prime}\int\underline{B}_{2} \qquad \textbf{(78)}$$

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